

Coarse Spaces Based on Higher-Order Interpolation for Schwarz Preconditioners for Helmholtz Problems

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1 Introduction

The Helmholtz equation, though seemingly simple, presents significant numerical challenges, particularly at large wavenumbers. These challenges can be attributed to two main problems. Firstly, a pollution error, a type of numerical dispersion error due to discrepancies between actual and numerical wavenumbers, necessitates grid refinement as the wavenumber increases, enlarging the linear system [4]. The second problem is the slow convergence behavior of iterative solvers for large wavenumbers and matrix dimensions.

Despite extensive research into numerical solvers for Helmholtz problems, no scalable and wavenumber-robust solver has yet been developed; the situation becomes even more difficult when considering heterogeneities or the elastic Helmholtz equation.

While the convergence of the conjugate gradient (CG) method for symmetric positive definite problems can be bounded via the condition number of the preconditioned matrix, $\kappa(M^{-1}A)$, the convergence is much less understood for solving indefinite problems using, for instance, the generalized minimal residual (GMRES) method [13]. For GMRES, if the coefficient matrix A is non-normal but diagonalizable, its convergence can resemble that of a normal matrix. Thus, the convergence rate cannot be directly bounded via the eigenvalues. However, they significantly influence it; cf. [6].

Recently, in [6, 7], higher-order Bézier deflation vectors have been employed to design nearly wavenumber-independent multigrid solvers. This is facilitated by the use of a higher-order coarse interpolation. Moreover, there has also been work in

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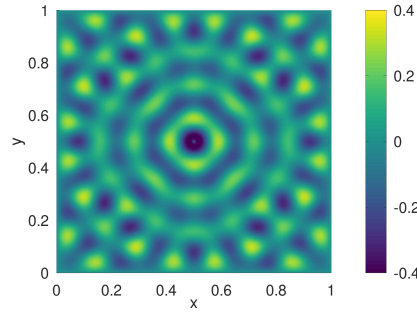


Fig. 1 Analytical solution of MP-1 with $k = 50$ on the unit square domain with homogeneous Dirichlet boundary conditions and the point source term located at the centre of the domain.

Schwarz domain decomposition preconditioners for Helmholtz problems with high wavenumbers; see, for instance, [8, 9, 10, 1, 11].

In this work, we employ coarse spaces for two-level Schwarz preconditioners spanned by higher-order Bézier functions. We observe good convergence properties for high wavenumber cases and increasing numbers of subdomains.

2 Problem Description

We consider the two-dimensional Helmholtz equations on the computational domain $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, with constant wavenumber $k > 0$ and either a Dirichlet boundary condition or a Sommerfeld radiation condition. As the first model problem, we seek the unknown field $u(x, y)$ that satisfies

$$\begin{cases} -\nabla u(x, y) - k^2 u(x, y) = \delta(x - \frac{1}{2}, y - \frac{1}{2}), & (x, y) \in \Omega, \\ u(x, y) = 0 & \text{for } (x, y) \in \partial\Omega. \end{cases} \quad (1)$$

A standard Dirac delta function is used as the point source term. We refer to this boundary value problem (BVP) as MP-1, and its analytical solution is depicted in Fig. 1. For the second model problem, we replace the Dirichlet boundary condition with the Sommerfeld radiation condition given by

$$\left(\frac{\partial u(x, y)}{\partial \mathbf{n}} - iku(x, y) \right) = 0, \quad \text{for } (x, y) \in \partial\Omega,$$

where \mathbf{n} is the outward pointing normal unit vector. We will refer to the model problem with Sommerfeld radiation condition as MP-2.

We use a uniform structured grid G_h , where h in G_h denotes the mesh size $h = 1/(n - 1)$; and n is the number of grid nodes in one dimension. Then, we discretize using a second-order finite difference scheme, resulting in a linear system

of equations

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad (2)$$

For both MP-1 and MP-2, the matrix A is sparse and symmetric. However, when using Dirichlet boundary conditions the linear system is indefinite real symmetric, while using the Sommerfeld radiation conditions results in an indefinite complex symmetric but non-Hermitian system. We consider both model problems because MP-2 resembles real-life infinite domain wave propagation more, while MP-1 is more difficult for iterative solvers due to the absence of damping.

One major problem in numerically solving Helmholtz problems is the pollution error, which is a type of numerical dispersion error that accumulates due to discrepancies between the actual and numerical wavenumbers. For second-order finite difference discretizations, avoiding this error requires grid refinement with $k^3 h^2 \leq 1$ [4]. However, this leads to very large linear systems as k increases. Therefore, the lighter condition $\kappa_h := kh \leq C$ is often used, implying a fixed number of grid nodes per wavelength λ . The coarse grid in the two-level Schwarz preconditioners should also be sufficiently refined for increasing wavenumbers. We use $\kappa_H := kH = 1$ [9, 10], with coarse grid size H . Another way of reducing the pollution error is to adopt Isogeometric Analysis as a spatial discretization technique [5].

3 Two-Level Schwarz Preconditioners

It has been shown that the classical one-level Schwarz domain decomposition methods are not scalable for Helmholtz problems; cf. [9, 10].

In this research, we consider two-level Schwarz preconditioners which are constructed from a nonoverlapping domain decomposition

$$\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$$

as follows: On the first level, we extend the subdomains by layers of grid nodes to construct overlapping subdomains $\Omega_1, \dots, \Omega_N$. Then, R_i corresponds to the discrete restriction from Ω to Ω_i , for $i = 1, \dots, N$. On the second level, we define a coarse space given by coarse basis functions being the rows of R_0 , which are defined on a coarse mesh G_H with the same step size as the subdomain size H .

We consider different types of two-level overlapping Schwarz preconditioners, beginning with the two-level additive Schwarz (AS2) preconditioner given by

$$M_{\text{AS2}}^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^N R_i^T (R_i A R_i^T)^{-1} R_i. \quad (3)$$

This corresponds to the case of exact local and coarse solvers, meaning that the local and the coarse problems are solved using a sparse direct solver. For the Laplace problem and standard Lagrangian coarse basis functions, an upper bound for the condition number of the preconditioned system matrix is then given by

$$\kappa(M_{\text{AS2}}^{-1}A) \leq C \left(1 + \frac{H}{\delta}\right); \quad (4)$$

cf. [15, Theorem 3.13, p. 69].

This indicates numerical scalability, as the condition number is bounded by a constant when increasing the number of subdomains, as long as the ratio of H/δ is kept fixed. Although we do not have a theoretical proof, we will analyze the numerical scalability of two-level Schwarz preconditioners with the coarse spaces to be described in Section 4 for Helmholtz problems in numerical experiments.

In addition to the AS2 preconditioner defined in eq. (3), we also consider the two-level scaled additive Schwarz (SAS2) preconditioner

$$M_{\text{SAS2}}^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^N R_i^T D_i (R_i A R_i^T)^{-1} R_i, \quad (5)$$

with the diagonal matrices D_i satisfying $\sum_{i=1}^N R_i^T D_i R_i = I$; cf. [3]. Furthermore, inspired by the use of deflation techniques for preconditioning Helmholtz problems in [6, 14, 9, 11], we also consider the two-level scaled hybrid Schwarz (SHS2) preconditioner, defined as

$$M_{\text{SHS2}}^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0 + \left(\sum_{i=1}^N R_i^T D_i (R_i A R_i^T)^{-1} R_i \right) \left(I - A \left(R_0^T (R_0 A R_0^T)^{-1} R_0 \right) \right), \quad (6)$$

where the projection $P_0 := (I - A (R_0^T (R_0 A R_0^T)^{-1} R_0))$ deflates out the coarse space.

Expression (6) is a combination of eq. (5) and the two-level hybrid Schwarz preconditioner from [2], which uses an ‘‘adapted deflation technique’’. The deflation should project the low frequency eigenvalues to zero and therefore remove the effect of those modes on the convergence of the iterative method.

4 Coarse Space

We define the coarse basis using higher-order Bézier interpolation on the coarse mesh taken from [6, 7] instead of classical linear coarse basis function; cf. [12, 15]. For the remainder of the article, the use of a linear coarse basis will be denoted as \mathbb{P}_1^H , where H is the coarse mesh size.

The higher-order interpolation functions based on second-order rational Bézier curves from [6, 7], for $H = 2h$, are given by

$$R_0 [u_h]_i = \frac{1}{8} \left([u_h]_{(m-2)} + 4 [u_h]_{(m-1)} + 6 [u_h]_m + 4 [u_h]_{(m+1)} + [u_h]_{(m+2)} \right), \quad (7)$$

for $i = 1, \dots, \frac{(n+1)}{2}$ and $m = 2i - 1$. This function for the restriction operator in eq. (7) is partially visualized in Fig. 2. The prolongation from the coarse to the fine mesh is then given by the transpose R_0^T . The use of these higher-order Bézier interpolation functions in the preconditioners is denoted by \mathbb{B}_2^H , where H is again the coarse mesh size. Note that due to this implementation, the size of the coarse matrix is the same for both the linear and the higher-order Bézier interpolation function. However, the coarse matrix following from the higher-order Bézier interpolation function has a lower sparseness when compared to using the linear interpolation.

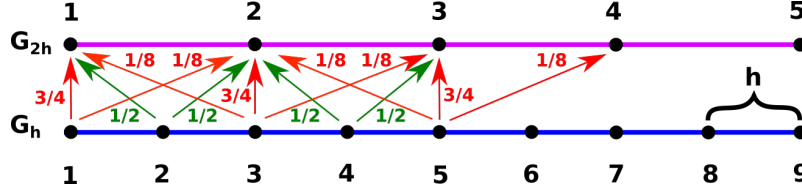


Fig. 2 Higher-order Bézier interpolation function for a fine and coarse grid with $H = 2h$.

5 Numerical Experiments

In our numerical experiments, we test the performance of the two-level Schwarz preconditioners described in Section 3 using the Bézier coarse interpolation defined in Section 4 for the Helmholtz model problems MP-1 and MP2. We also compare the performance against using linear coarse basis functions. Therefore, we employ the GMRES method with a relative tolerance of 10^{-7} and a maximum iteration cap of 100 for the first two tables and 50 for the others. These maximum iteration caps are on the low end, but since we are interested in wave-number independence having these maximum iteration caps is appropriate. In our tables, x indicates that the relative tolerance has not been reached within the maximum number of iterations. The implementation is done in MATLAB, and the computations have been carried out on a Intel Core i7-8557U processor with four CPU cores and 16 GB memory.

# subdomains	$ G_h $	/	$ G_H $	$\frac{1}{h}$	$\frac{1}{H}$	k	AS _{4h}	SAS _{4h}	SHS _{4h}	AS _{4h}	SAS _{4h}	SHS _{4h}
400	6 561	/	441	80	20	20	21	15	9	21	16	8
1 600	25 921	/	1 681	160	40	40	25	20	11	21	16	8
3 600	58 081	/	3 721	240	60	60	29	25	14	21	16	8
6 400	103 041	/	6 561	320	80	80	34	30	16	21	16	8
10 000	160 801	/	10 201	400	100	100	40	35	19	21	16	8
14 400	231 361	/	14 641	480	120	120	45	41	22	21	16	8
19 600	314 721	/	19 881	560	140	140	52	48	26	21	16	8
40 000	641 601	/	40 401	800	200	200	75	71	38	21	16	8

Table 1 Number of iterations results for preconditioners using \mathbb{P}_1 (middle columns) or \mathbb{B}_2 (right columns) interpolation for MP-2, with $\kappa_h = 0.25$, $H_{sub} = H = 4h$.

# subdomains	$ G_h $	/	$ G_H $	$\frac{1}{h}$	$\frac{1}{H}$	k	AS _{4h}	SAS _{4h}	SHS _{4h}	AS _{4h}	SAS _{4h}	SHS _{4h}
400	6 561	/	441	80	20	20	21	16	9	20	14	7
1 600	25 921	/	1 681	160	40	40	40	34	18	18	13	6
3 600	58 081	/	3 721	240	60	60	55	49	26	19	13	7
6 400	103 041	/	6 561	320	80	80	x	x	53	15	12	5
10 000	160 801	/	10 201	400	100	100	x	x	55	18	13	6
14 400	231 361	/	14 641	480	120	120	x	x	89	19	13	7
19 600	314 721	/	19 881	560	140	140	x	x	x	18	12	6
40 000	641 601	/	40 401	800	200	200	x	x	x	19	13	7

Table 2 Number of iterations results for preconditioners using \mathbb{P}_1 (middle columns) or \mathbb{B}_2 (right columns) interpolation for MP-1, with $\kappa_h = 0.25$, $H_{sub} = H = 4h$. x denotes the maximum number of iterations of 100 being reached. Maximum subdomain overlap.

Scalability of \mathbb{P}_1 and \mathbb{B}_2 coarse spaces For the numerical experiments, we use $H = 4h$, $\kappa_h = 0.25$, and maximize the subdomain overlap δ . The overlap is increased such that any grid node belongs to at most 4 subdomains.

In Table 1, we compare iteration counts for MP-2 for linear coarse interpolation (middle columns) and higher-order Bézier grid interpolation (right columns). As anticipated, for \mathbb{P}_1 , the number of iterations required to reach convergence increases with the wavenumber, whereas the convergence is robust with respect to the wavenumber when using higher-order Bézier grid interpolation. The iteration count is the lowest for the SHS_{4h} preconditioner, which requires to first solve the coarse problem before solving the local problems.

Next, in Table 2, the corresponding results for MP-1 are listed. The results are similar to those for MP-2 in that the iteration count for the \mathbb{P}_1 coarse space increases with the wavenumber, and the convergence for \mathbb{B}_2 coarse space is robust.

The subdomain sizes are relatively low, and hence, the dimension of the coarse space, $|G_H|$, is relatively high compared with the original problem size, $|G_h|$. Hence, in parallel computations, the solution of the coarse problem would quickly become a bottleneck. In distributed-memory parallel computations, we would typically have

$k \setminus n$	33	41	49	57	65	73	81	89	97	105	113	121	129	137	145	153	161
10	7	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5
20	18	17	11	9	8	7	7	7	7	7	7	6	6	6	6	6	6
30	38	23	32	37	17	11	9	8	7	7	7	6	6	6	6	6	6
40	x	x	45	x	x	x	43	15	10	9	8	7	7	7	7	6	6
50	x	x	x	x	48	x	x	x	x	31	14	11	9	9	8	8	7

Table 3 Number of iterations with the $\text{SHS}_{4h}/\mathbb{B}_2^{4h}$ preconditioner for MP-1 for small k . x denotes the maximum NOI of 50 being reached. Maximum subdomain overlap.

$k \setminus n$	33	49	65	81	97	113	129	145	161	177	193	209	225	241	257
5	5	8	7	8	7	8	7	8	7	8	7	8	7	8	7
10	6	11	10	13	9	8	7	7	7	7	7	7	7	7	7
15	11	11	12	17	12	16	12	12	9	10	9	9	8	9	8
20	7	13	14	18	16	20	18	x	21	17	12	11	9	10	9
25	8	16	21	26	18	45	19	28	24	31	24	23	14	13	10
30	9	22	33	28	39	38	49	36	33	x	34	40	40	29	18

Table 4 Number of iterations with the $\text{SHS}_{16h}/\mathbb{B}_2^{16h}$ preconditioner for MP-1 for small k . x denotes the maximum NOI of 50 being reached. Maximum subdomain overlap.

much more memory available on each rank, such that we could significantly increase the subdomain size, resulting in a better ratio of the coarse and fine grid sizes.

Detailed variation of k and n Finally, we further analyze the performance of the best-performing preconditioner, $\text{SHS}_H/\mathbb{B}_2^H$ by varying the wavenumber k and the grid size n . In particular, we investigate the limit of the wavenumber robustness of the preconditioner for MP-1. We vary the coarsening ratio H/h between 4 and 16, as shown in Tables 3 and 4. As expected, if n is too low, the iteration count increases with the wavenumber. This is due to h and H being too large to maintain suitable κ_h and κ_H ratios.

Looking at Table 3, we observe that for $H = 4h$, $\kappa_h = 0.25$ seems to be suitable. From Table 4 on the other hand, with $H = 16h$, $\kappa_h = 0.0625$ seems to be required for wavenumber robustness, which is $1/4$ of that for $H = 4h$. This seems to indicate that the wavenumber robustness also requires sufficient resolution of the coarse mesh rather than only for the fine mesh. In particular, the same $\kappa_H := kH \leq 1$ seems to be required for both subdomain sizes.

6 Conclusion

We have shown that using higher-order Bézier interpolation for the coarse space in two-level Schwarz preconditioners can yield scalable and wavenumber robust coarse spaces for Helmholtz problems. In particular, wavenumber robustness can be obtained for cases where a linear coarse space does not yield robust convergence; nonetheless, we observed that a certain refinement level of the coarse mesh relative to the wavenumber is required.

The best-performing preconditioner is the two-level scaled hybrid Schwarz preconditioner, which uses a higher-order approximation scheme to construct the deflation vectors for the coarse space.

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