Convergence of Neumann-Neumann Waveform Relaxation Algorithm with Time-dependent Relaxation Parameter for Time-fractional Sub-diffusion and Diffusion-wave Problems

Bankim C. Mandal[0009−0009−0134−0422] 1 and Soura Sana[0000−0003−1892−9486] 1

1 Introduction

As the realistic modeling of complex phenomena progresses, so does the application of fractional calculus [9]. Fractional calculus, pioneered nearly 300 years ago by Riemann, Caputo, and others, becomes particularly significant.

The increasing computational complexity drives the creation of new methods to leverage the multi-core structure of modern supercomputers. One such method is the Domain Decomposition algorithm, coupled with Waveform Relaxation. Variants such as Neumann-Neumann (NNWR), Dirichlet-Neumann (DNWR), Schwarz (SWR), and Optimized Schwarz (OSWR) methods have been developed [1, 2, 5, 7, 3]. These techniques typically exhibit fast convergence for small time windows, prompting the introduction of pipeline and adaptive pipeline methods [8, 6]. The convergence of DNWR, NNWR, and OSWR methods relies heavily on the choice of relaxation parameter. In this article, we opt for a time-dependent relaxation parameter for the analysis of FPDEs, specifically sub-diffusion and diffusion-wave equations. Hence, we take the following problem as our guiding example:

$$\begin{cases} {}_0D_t^{\beta} u = \nabla \cdot (\mu(\boldsymbol{x})\nabla u) + f(t,\boldsymbol{x}), & (0,T)\times\Omega, \\ u(t,\boldsymbol{x}) = h(t,\boldsymbol{x}), & (0,T)\times\partial\Omega, \\ u(0,\boldsymbol{x}) = g_0(\boldsymbol{x}), & \Omega, \\ \partial_t u(t,\boldsymbol{x})|_{t=0} = \omega_0(\boldsymbol{x}), & \Omega, \quad \text{when } \beta \geq 1. \end{cases} \tag{1}$$

Here, $\mu(\boldsymbol{x}) > 0$ is the diffusion coefficient, and f, h, g_0, ω_0 are sufficiently smooth and ${}_0D_t^{\zeta}$ is the Caputo fractional derivative defined for order ζ, with $p = \lceil \zeta \rceil$ as follows:

Bankim C. Mandal and Soura Sana
School of Basic Sciences, Indian Institute of Technology Bhubaneswar,
e-mail: {bmandal, ss87}@iitbbs.ac.in

$$ {}_0D_t^{\zeta} x(s) := \frac{1}{\Gamma(p-\zeta)} \int_0^s (s-\xi)^{p-\zeta-1} x^{(p)}(\xi) d\xi. $$

2 NNWR for Time-Fractional PDEs

We are considering the NNWR algorithm for a 1D domain denoted as Ω. The domain is divided into N non-overlapping sub-domains, such that there are no cross points, formally expressed as $\Omega = \cup_1^N \bar{\Omega}_j$, and $\Omega_i \cap \Omega_j = \emptyset$. We assume different generalized diffusion parameter $\mu = \mu_i$ on Ω_i. For our initial guesses $p_{i,j}^{(k)}$, we focus on the artificial boundary $(0,T) \times (\partial\Omega_i \cap \partial\Omega_j)$, where $j = i-1, i+1$. Here, $\boldsymbol{n}_{i,j}$ represents the outward unit normal drawn on $\partial\Omega i \cap \partial\Omega_j$ from the i-th to the j-th direction. Given this context, the NNWR algorithm can be expressed as follows:

We solve for $k = 1, 2, \cdots$

$$ \begin{cases} {}_0D_t^{\beta} u_i^{(k)} = \nabla \cdot \left(\mu(\boldsymbol{x}) \nabla u_i^{(k)}\right) + f, & (0,T) \times \Omega_i, \\ u_i^{(k)} = h, & (0,T) \times (\partial\Omega_i \cap \partial\Omega), \\ u_i^{(k)} = p_{i,j}^{(k-1)}, & (0,T) \times (\partial\Omega_i \cap \partial\Omega_j), \\ u_i^{(k)}(\boldsymbol{x}, 0) = g_0(\boldsymbol{x}), & \Omega_i, \\ \partial_t u_i^{(k)}(t, \boldsymbol{x})|_{t=0} = \omega_0(\boldsymbol{x}), & \Omega_i, \quad \text{when } \beta > 1, \end{cases} \tag{2} $$

and then solve simultaneous Neumann problems across all subdomains,

$$ \begin{cases} {}_0D_t^{\beta} \psi_i^{(k)} = \nabla \cdot \left(\mu(\boldsymbol{x}) \nabla \psi_i^{(k)}\right), & (0,T) \times \Omega_i, \\ \psi_i^{(k)} = 0, & (0,T) \times (\partial\Omega_i \cap \partial\Omega), \\ \mu_i \partial_{\boldsymbol{n}_{i,j}} \psi_i^{(k)} = \mu_i \partial_{\boldsymbol{n}_{i,j}} u_i^{(k)} + \mu_j \partial_{\boldsymbol{n}_{j,i}} u_j^{(k)}, & (0,T) \times (\partial\Omega_i \cap \partial\Omega_j), \\ \psi_i^{(k)}(\boldsymbol{x}, 0) = 0, & \Omega_i, \\ \partial_t \psi_i^{(k)}(t, \boldsymbol{x})|_{t=0} = 0, & \Omega_i, \quad \text{when } \beta > 1. \end{cases} \tag{3} $$

The following formula is subsequently employed to update the interface values

$$ p_{i,j}^{(k)}(t, \boldsymbol{x}) = p_{i,j}^{(k-1)}(t, \boldsymbol{x}) - \theta_{i,j} \left(\psi_i^{(k)} \big|_{(0,T)\times\Omega_{i,j}} + \psi_j^{(k)} \big|_{(0,T)\times\Omega_{i,j}} \right). \tag{4} $$

Our objective is to investigate the convergence of error terms at artificial boundaries i.e. $\upsilon_{i,j}^{(k-1)}(t, \boldsymbol{x}) := u_i|_{(0,T)\times\Omega_{i,j}} - p_{i,j}^{(k-1)}(t, \boldsymbol{x})$ as $k \to \infty$. The convergence result is analyzed and presented in [10, 11].

3 NNWR Analysis

We will now derive the expression of the update condition for the NNWR method with a time-variable parameter. Our prerequisite is that $\theta(t)$ should be any real-valued polynomial for $t \in (0,T)$, specifically expressed as $\theta(t) = \sum_{i=0}^{m} c_i t^i$, $m \in \mathbb{N}$. To conduct our analysis, we confine ourselves to a one-dimensional domain, comprising only two subdomains: the first subdomain spans the interval $(-l_1, 0)$, and the second subdomain spans $(0, l_2)$. The solution of the Dirichlet part (1.2) of the NNWR algorithm for the error equations in Laplace space is obtained as follows:

$$\hat{u}_1^{(k)}(x,s) = \hat{p}^{(k-1)}(s)\,\mathrm{csch}\left(l_1\sqrt{s^\beta/\mu_1}\right)\sinh\left\{(x+l_1)\sqrt{s^\beta/\mu_1}\right\},$$
$$\hat{u}_2^{(k)}(x,s) = \hat{p}^{(k-1)}(s)\,\mathrm{csch}\left(l_2\sqrt{s^\beta/\mu_2}\right)\sinh\left\{(l_2-x)\sqrt{s^\beta/\mu_2}\right\},$$

and the solutions of part (1.3) are

$$\hat{\psi}_1^{(k)}(x,s) = \hat{r}^{(k)}(s)\left(s^\beta\mu_1\right)^{-1/2}\sinh\left\{(x+l_1)\sqrt{s^\beta/\mu_1}\right\}\mathrm{sech}\left(l_1\sqrt{s^\beta/\mu_1}\right), \quad (5)$$
$$\hat{\psi}_2^{(k)}(x,s) = \hat{r}^{(k)}(s)\left(s^\beta\mu_2\right)^{-1/2}\sinh\left\{(l_2-x)\sqrt{s^\beta/\mu_2}\right\}\mathrm{sech}\left(l_2\sqrt{s^\beta/\mu_2}\right), \quad (6)$$

where,

$$\hat{r}^{(k)}(s) = \hat{p}^{(k-1)}(s)\left[\sqrt{\mu_1 s^\beta}\coth(l_1\sqrt{s^\beta/\mu_1}) + \sqrt{\mu_2 s^\beta}\coth(l_2\sqrt{s^\beta/\mu_2})\right].$$

Utilizing the solution to the Neumann problem (1.5)-(1.6) in the update condition yields:

$$\hat{p}^{(k)}(s) = \hat{p}^{(k-1)}(s) - \sum_{i=0}^{m}(-1)^i c_i \frac{d^i}{ds^i}\left[\left(2+\sqrt{\mu_1/\mu_2}+\sqrt{\mu_2/\mu_1}\right)\hat{p}^{(k-1)}(s)\right]$$
$$-\sum_{i=0}^{m}(-1)^i c_i \frac{d^i}{ds^i}\left[(\sqrt{\mu_1/\mu_2}\hat{H}_1(s)+\sqrt{\mu_2/\mu_1}\hat{H}_2)(s)\hat{p}^{(k-1)}\right].$$

where $\hat{H}_1(s) = (\tanh(l_2\sqrt{s^\beta/\mu_2})\coth(l_1\sqrt{s^\beta/\mu_1}) - 1)$ and $\hat{H}_2(s) = (\tanh(l_1\sqrt{s^\beta/\mu_1})\coth(l_2\sqrt{s^\beta/\mu_2}) - 1)$. According to Lemma 6.1 and Lemma 6.2 in [10], we know that the inverse Laplace transforms of $\hat{H}_1(s)$ and $\hat{H}_2(s)$, denoted as $H_1(t)$ and $H_2(t)$ respectively, are well defined. Therefore, applying the inverse Laplace transform, yields:

$$\begin{aligned} p^{(k)}(t) &= p^{(k-1)}(t) - \sum_{i=0}^{m} c_i t^i \left[\left(2 + \sqrt{\mu_1/\mu_2} + \sqrt{\mu_2/\mu_1}\right) p^{(k-1)}(t)\right] \\ &\quad - \sum_{i=0}^{m} c_i t^i \int_0^t (\sqrt{\mu_1/\mu_2} H_1(t) + \sqrt{\mu_2/\mu_1} H_2(t))(t-\tau) p^{(k-1)}(\tau) d\tau \\ &= \sigma(t) p^{(k-1)}(t) - \theta(t) \int_0^t L(t-\tau) p^{(k-1)}(\tau) d\tau, \end{aligned} \tag{7}$$

with $\sigma(t) := \{1-(2+\sqrt{\mu_1/\mu_2}+\sqrt{\mu_2/\mu_1})\theta(t)\}, L(t) := \sqrt{\mu_1/\mu_2}H_1(t)+\sqrt{\mu_2/\mu_1}H_2(t)$.

Convergence Study: Note that any continuous function defined on a compact domain can be approximated arbitrarily closely by polynomials. Therefore one may include general continuous $\theta(t)$ in asymptotic sense. Our subsequent objective is to determine $\theta(t)$ in order to achieve faster convergence. In operator notation, the update component in the NNWR method described by equation (1.7) can be expressed as follows: $Rp(t) = Mp(t) - Qp(t)$, where

$$Mp(t) = \sigma(t)p(t) \quad \text{and} \quad Qp(t) = \theta(t) \int_0^t L(t-\tau)h(\tau)d\tau.$$

Once more, referring to Lemma 6.1 and Lemma 6.2 in [10], we establish the continuity of functions $\hat{H}_1$ and $\hat{H}_2$, as well as the continuity of $\theta(t)$ over the interval $[0, T]$. Consequently, the operators $M, Q : C([0, T]) \to C([0, T])$ are compact on $C([0, T])$. However, it is worth noting that the operator R can also be defined in alternative spaces. To accomplish this, we invoke the following result, also known as bounded linear transformation theorem, see [4].

Theorem 1. *Let A and B be Banach spaces, and let F be a linear operator from a dense subspace $\tilde{A}$ of A to B. Suppose there exists a constant ξ such that for all $y \in \tilde{A}$, we have $||Fy|| \le \xi||y||$. Then F can be uniquely extended to a bounded linear operator from A to B while maintaining the aforementioned estimate on A.*

The operator R can thus be extended to be a bounded linear operator on $L^2([0, T])$. We also have the operator Q quasi-nilpotent and compact on $L^2([0, T])$, implying that its spectrum $\lambda(Q) = \{0\}$, i.e. spectral radius $\rho(Q) = 0$. With this setup, we are now ready to demonstrate our main results.

Theorem 2 ($\theta(t)$ is constant)**.** *If $\theta(t)$ remains constant over a bounded time interval $[0, T]$, then having $\sigma(t) = 0$ results in a super-linear convergence rate.*

Proof. Let $\theta(t) = \theta$ be a constant function. Then the operators M and Q commute with each other. Thus, based on the properties of the spectral radius for commutative operators, we have the following inequalities:

$$\rho(M - Q) \le \rho(M) + \rho(-Q) \le \rho(M) + \rho(Q) = \rho(M),$$

which implies $\rho(R) \le \rho(M)$. Additionally,

$$\rho(M-Q) \geq |\rho(M) - \rho(Q)| = \rho(M),$$

yielding $\rho(R) \geq \rho(M)$. Combining these inequalities, we conclude that $\rho(R) = \rho(M)$. The convergence rate will be super-linear when $\rho(R) = 0$, i.e., when $\rho(M) = 0$. This establishes $\sigma(t) = 0$. □

Theorem 3 ($\theta(t)$ is continuous)**.** *If $\theta(t)$ is a continuous function over a bounded time interval $[0, T]$, then setting $\sigma(t) = 0$ results in the super-linear convergence.*

Proof. We consider the operator $Q : L^2([0,T]) \to L^2([0,T])$ to be compact, and we have $\sigma(t) \in C([0,T])$. Following the observation made in the remarks section after Theorem 3.3 in [12], we ascertain that the spectrum $\lambda(P)$ is equivalent to the range of $\sigma(t)$. Consequently, the operator R achieves super-linear convergence when $\sigma(t) = 0$. □

4 Numerical Illustration

Consider the model problem (1.1) with the initial condition given by $g_0(x) = x(2-x)\exp(-(x-1.1)^2)$, where the left boundary condition is defined as $h_l(0,t) = t\exp(-3t)$ and the right boundary condition is given by $h_r(2,t) = t^3\exp(-2t)$. We select a zero forcing term. We consider the domain $\Omega = (0,2)$ is to be divided into $\Omega_1 = (0, 1.2)$ and $\Omega_2 = (1.2, 2)$. The time window is set to $T = 1$, with a time discretization $\Delta t = 0.01$.

In Figure 1.1, we compare the errors after 10 iterations as the relaxation parameter θ varies in the range $(0, 0.5)$. For this experiment, we set the diffusion parameter for the first sub-domain to $\mu_1 = 0.25$ and for the second sub-domain to $\mu_2 = 1$. We observe from this experiment that for $\theta = \left(2 + \sqrt{\frac{\mu_1}{\mu_2}} + \sqrt{\frac{\mu_2}{\mu_1}}\right)^{-1} = 0.22$, we achieve the best convergence rate for both sub-diffusion and diffusion-wave problems. This observation aligns perfectly with our analysis in Theorem 2, which demonstrates that setting $\sigma(t) = 0$ results in super-linear convergence, while other values of θ generally lead to linear convergence estimates.

In Figure 1.2, we present the contour showing how the error varies after 10 iterations for different values of c_0 and c_1, where c_0 and c_1 represent the coefficients of the linear function $\theta(t) = c_0 + c_1 t$. The experiments maintain the same diffusion parameter and sub-domain size as in previous trials. Our observations indicate that the most favorable convergence occurs when $c_0 = 0.22$ and $c_1 = 0$, regardless of the fractional order, β, being 0.5 or 1.5. This finding aligns closely with the analysis presented in Theorem 3, which suggests that super-linear convergence is attained when $\sigma(t) = 0$.

In Figure 1.3, we compare the errors after 10 iterations while varying the relaxation parameters θ_1 and θ_2 within the range $(0, 0.5)$. For this experiment, we set the diffusion parameter for the first sub-domain as $\mu_1 = 0.25$, for the second sub-domain as $\mu_2 = 1$, and for the third sub-domain as $\mu_3 = 0.25$, with subdomains defined as

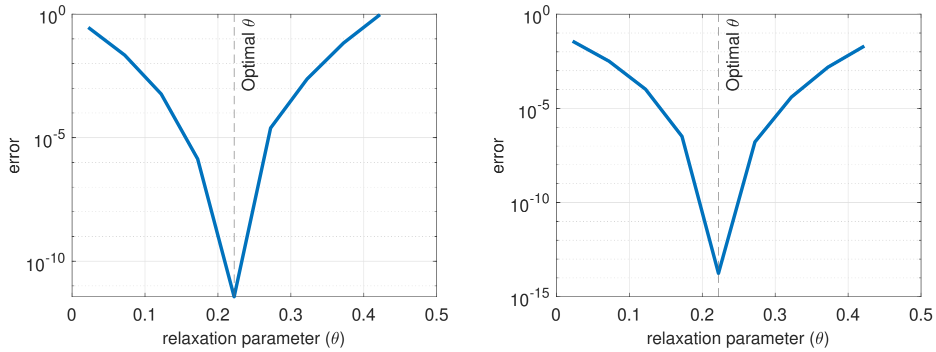

Fig. 1: The convergence rate of the NNWR algorithm after ten iterations with two subdomains, having diffusion coefficients of $\mu_1 = 0.25$ and $\mu_2 = 1$, respectively, is examined for different values of the relaxation parameter θ at time $T = 1$. The fractional order is set to $\beta = 0.5$ on the left and $\beta = 1.5$ on the right.

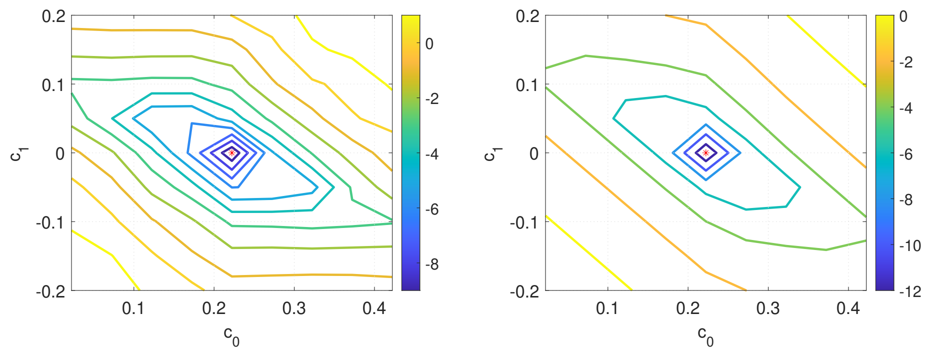

Fig. 2: The convergence rate of the NNWR algorithm after ten iterations is examined with two subdomains having diffusion coefficients $\mu_1 = 0.25$ and $\mu_2 = 1$. This examination is carried out for various values of the linear relaxation parameter $\theta(t) = c_0 + c_1 t$ at time $T = 1$. The fractional order on the left is $\beta = 0.5$, while on the right it is $\beta = 1.5$.

$\Omega_1 = (0, 0.5)$, $\Omega_2 = (0.5, 1.5)$, and $\Omega_3 = (1.5, 2)$. From this experiment, we observe that by setting $\theta_j = \left(2 + \sqrt{\frac{\mu_j}{\mu_{j+1}}} + \sqrt{\frac{\mu_{j+1}}{\mu_j}}\right)^{-1} = 0.22$ for $j = 1, 2$, we achieve the best convergence rates for both sub-diffusion and diffusion-wave problems.

Using the similar analytical technique one can show that for DNWR with two sub-domains, $\theta = \left(1 + \sqrt{\frac{\mu_1}{\mu_2}}\right)^{-1}$ yields a super-linear convergence rate. In the forthcoming series of numerical experiments, we aim to validate this assertion. We maintain identical sub-domain sizes and diffusion coefficients as those employed in the two-

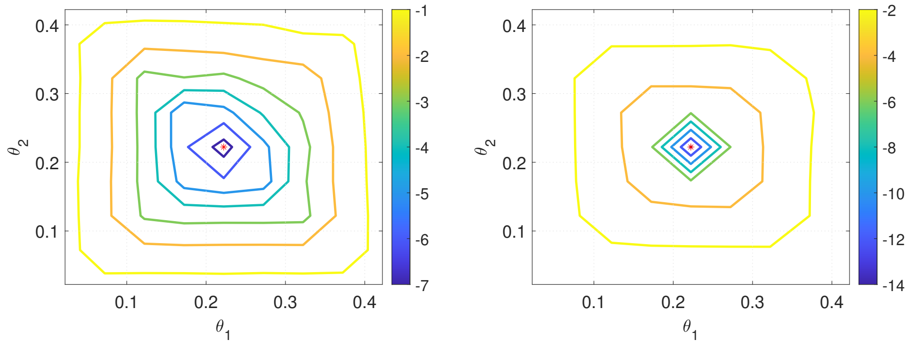

Fig. 3: The convergence rate of the NNWR algorithm after ten iterations for three subdomains with diffusion coefficients $\mu_1 = 0.25$, $\mu_2 = 1$, and $\mu_3 = 0.25$, considering various constant values of relaxation parameters θ_1 and θ_2 at time $T = 1$. The fractional order on the left is $\beta = 0.5$, and on the right is $\beta = 1.5$.

subdomain scenario for the NNWR algorithm. As depicted in Figure 1.4, we observe that $\theta = \left(1 + \sqrt{\frac{\mu_1}{\mu_2}}\right)^{-1} = 0.66$ offers the most favorable convergence rate.

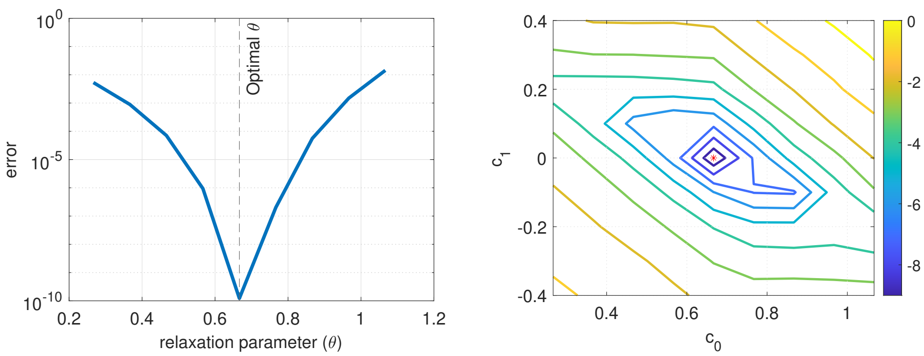

Fig. 4: The convergence rate of the DNWR algorithm after ten iterations with two subdomains is analyzed for different values of the relaxation parameter θ at time $T = 1$ in the context of $1D$ sub-diffusion with a fractional order of $\beta = 0.5$. Specifically, we consider two cases: on the left, $\theta(t) = c_0$, and on the right, $\theta(t) = c_0 + c_1 t$.

Now, we extend our experiments to 2D time-fractional Allen-Cahn equation which is as follows:

$$\begin{cases} {}_0D_t^{\beta} u = \mu \nabla \cdot (\nabla u) + (u - u^3), & (0,T) \times \Omega, \\ u(t,x,y) = 0, & (0,T) \times \partial\Omega, \\ u(0,\boldsymbol{x}) = \sin(2\pi x)\sin(\pi y), & \Omega, \\ \partial_t u(t,\boldsymbol{x})|_{t=0} = 0, & \Omega, \quad \text{when } \beta > 1. \end{cases}$$

Here, the domain $\Omega = (0,2)\times(-1,1)$ is divided into two subdomains of equal length. The diffusion term in the equation is scaled by the parameter $\mu_1 = \mu_2 = \mu = 0.0001$ for both subdomains, with a fractional order of $\beta = 0.5$. In Figure 1.5, we compare the convergence rates after 10 iterations with various values of the relaxation parameter. In the left sub-plot, we utilize a constant relaxation parameter and observe that the best convergence rate is achieved at $\theta = \left(1 + \sqrt{\frac{\mu_1}{\mu_2}}\right)^{-1} = 0.5$. In the right sub-plot, we employ a linear relaxation parameter $\theta(t)$. Again, we notice that the best convergence rate is achieved when $\theta = 0.5$.

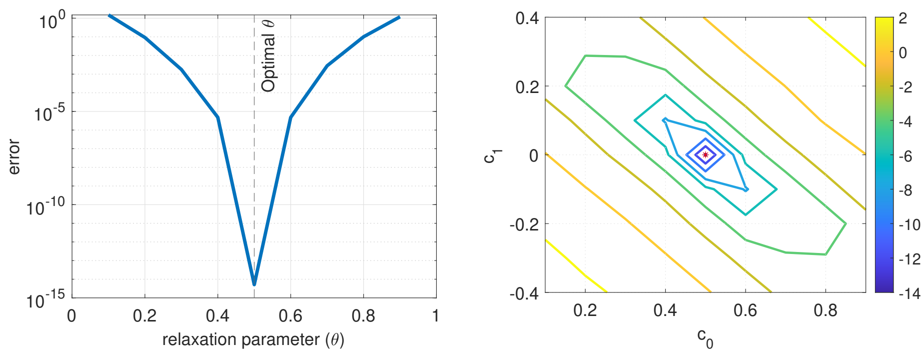

Fig. 5: The convergence rate of the DNWR algorithm after ten iterations with two subdomains is examined for different values of the relaxation parameter θ at time $T = 1$ for the 2D Allen-Cahn equation with a fractional order of $\beta = 0.5$. On the left, we consider $\theta(t) = c_0$, while on the right, we explore $\theta(t) = c_0 + c_1 t$.

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