Elimination Strategies for Nonlinear Preconditioning of Incompressible Navier-Stokes

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1 Introduction

Many nonlinear preconditioning techniques—additive and multiplicative, overlapping and nonoverlapping, and left- and right-sided—have been demonstrated and undergirded theoretically [1, 4, 23]. Common to all, and so far without theoretical guidance, is the challenge of identifying the degrees of freedom and corresponding equations that impede the progress of Newton's method applied to the global system. Heuristic approaches based on residual magnitudes, on degrees of nonlinearity in the equations, on solution features (e.g., shocks, fronts, etc.), and on principal component analysis have been successful in different contexts. We present a hybrid of two strategies that have proved successful for incompressible Navier-Stokes: field-based and pointwise residual-based. We evaluate it on the classic problem of steady laminar flow behind a backward-facing step, using the right-preconditioned inexact Newton method with backtracking based on nonlinear elimination (INB-NE) method and show its advantages of robustness and convergence over either strategy alone.

Newton's method and its variants are widely employed for solving large-scale nonlinear systems arising from the discretization of partial differential equations, such as incompressible Navier-Stokes. When the system is provided with a sufficiently accurate initial guess lying within a roughly hyperspherical domain around the root, these methods typically demonstrate superlinear or quadratic convergence rates. However, they may be frustrated by "nonlinear stiffness" arising from highly

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distorted residual landscapes, which results in stagnation of residual norms or even failure of Newton's method.

To address the challenge of such unbalanced nonlinearities and enhance global convergence, various continuation techniques have been developed to obtain suitable initial iterates. These include parameter continuation [19], which gradually adjusts physical parameters (e.g., loadings or boundary conditions); mesh sequencing [32], where solutions are first computed on coarser grids before refinement; and pseudo-time-stepping [8], which introduces an artificial temporal evolution with small timesteps to approach steady-state solutions. These methods systematically solve a sequence of modified problems—whether through relaxed physical conditions, simplified discretizations, or controlled temporal integration—to progressively guide the solver toward the desired solution of the original nonlinear system. Nonlinear preconditioning provides an alternative approach to enhance the robustness of Newton-type methods by reducing their sensitivity to initial guesses and fundamentally improving the nonlinear conditioning of the system. When necessary, this strategy can be effectively combined with other globalization techniques, such as those previously discussed, to further strengthen convergence behavior.

In analogy to linear preconditioning, nonlinear preconditioning techniques handle unbalanced nonlinearities through either left- or right-side nonlinear transformations. Classical left nonlinear preconditioners reformulate the original system into an equivalent modified system that preserves the solution while exhibiting improved nonlinear convergence properties. This preconditioned system is then solved using an outer Jacobian-free Newton method [19]. Examples include the additive and multiplicative Schwarz preconditioned inexact Newton methods, ASPIN [1, 3, 4, 15, 31] and two-level ASPIN [5, 28], MSPIN [25] and its multiple variants [21, 23, 33], as well as the restricted nonlinear Schwarz preconditioners RASPEN [9, 13] and SRASPEN [7].

In contrast, right nonlinear preconditioners recast the basis for the solution space by a nonlinear transformation and find the solution in the transformed space rather than in the original space. The right nonlinear preconditioners are often associated with a nonlinear elimination (NE) [20] procedure. Nonlinear elimination (NE) [20] handles nonlinear "imbalance" by strategically eliminating strongly nonlinear variables before the global Newton solve. The nonlinear elimination preconditioned inexact Newton methods [12, 14, 34] have been applied effectively to such challenging problems as incompressible Navier-Stokes equations at high Reynolds numbers [26], blood flow in branching arteries [27], and two-phase flow in porous media [35]. There is also growing interest in nonlinear FETI (finite element tearing and interconnecting) [29], nonlinear FETI-DP (FETI-dual primal) and nonlinear BDDC (balancing domain decomposition by constraints) [16, 17, 18], further expanding the family of right nonlinear preconditioners.

The computational efficiency of the nonlinear elimination preconditioned inexact Newton methods—including the INB-NE approach—are highly dependent on the selection of eliminated variables (the so-called "bad" components). Herein, we implement a right-preconditioning approach, INB-NE, for resolving flow over a backward-facing step, and compare the effects of three variable elimination strategies on the method's convergence.

2 Backward-facing Step Flow

We consider the backward-facing step flow [10, 11] within a channel defined on $\Omega = (0, 30) \times (-0.5, 0.5)$ with three unknowns: the velocity fields (u, v) in the (x, y) directions and the vorticity ω. The governing equations consist of the nondimensional steady-state Navier-Stokes equations in vorticity-velocity form

$$\begin{cases} -\Delta u - \frac{\partial \omega}{\partial y} = 0, \\ -\Delta v + \frac{\partial \omega}{\partial x} = 0, \\ -\frac{1}{Re}\Delta \omega + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0, \end{cases} \tag{1}$$

where Re denotes the Reynolds number. We close the system by imposing the following boundary conditions:

- Along the left boundary Γ_{left},

$$\begin{cases} u = 24y(0.5 - y), \quad v = 0, \; y \in [0, 0.5], \\ u = 0, \quad v = 0, \qquad\qquad y \in [-0.5, 0), \end{cases} \qquad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the top boundary Γ_{top},

$$u = 0, \quad v = 0, \quad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the bottom boundary Γ_{bottom},

$$u = 0, \quad v = 0, \quad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the right boundary Γ_{right},

$$u_x = 0, \quad v_x = 0, \quad \omega_x = 0.$$

The computational domain Ω is discretized using a structured uniform grid of rectangular cells. We employ second-order central finite difference schemes to approximate both the Laplacian operators and first-order partial derivatives in equation (1). The vorticity ω is determined by its definition across all boundaries, with the exception of the outlet boundary Γ_{right} where we impose $\omega_x = 0$. For the left, top and bottom boundaries, we implement the vorticity boundary conditions using a second-order finite difference approximation that employs only immediately adjacent grid points, following the numerical treatment described in [30]. For the right boundary condition, a second-order backward finite difference scheme is employed to approximate the first-order derivatives u_x, v_x and ω_x.

Let N denote the total number of mesh points in the computational domain Ω. The unknown variables are arranged in a point-wise block ordering as

$$X = [u_0, v_0, \omega_0, u_1, v_1, \omega_1, \ldots, u_{N-1}, v_{N-1}, \omega_{N-1}]^T,$$

and the corresponding nonlinear system is structured in the same order:

$$F(X) = [F_0^u, F_0^v, F_0^\omega, F_1^u, F_1^v, F_1^\omega, \ldots, F_{N-1}^u, F_{N-1}^v, F_{N-1}^\omega]^T = 0,$$

where each component $F_i^* = F_i^*(X)$, for $* = u, v$ or ω, $i = 0, 1, \ldots, N-1$.

3 The INB-NE Algorithm

We consider a nonlinear system of algebraic equations $F : D \subset R^n \to R^n$, where we seek a vector $Y^* \in R^n$ such that

$$F(Y^*) = 0.$$

In contrast to left nonlinear preconditioning, which operates on the residual, right nonlinear preconditioning modifies the coordinates of the solution:

$$F(G(X)) = 0, \quad Y = G(X). \tag{2}$$

The operator G is typically defined implicitly through a nonlinear elimination process. First, we partition the index set $S = \{1, 2, \ldots, n\}$ into two disjoint subsets

$$S = S_b \bigcup S_g, \quad S_b \bigcap S_g = \emptyset,$$

and let n_1 and n_2 represent the dimensions of S_b and S_g, respectively, where $n_1 + n_2 = n$. Based on the partition, we define the subspaces of bad components and good components as

$$V_b = \{v | v = [v_1, \ldots, v_n]^T \in R^n, v_i = 0 \text{ if } i \notin S_b\},$$

$$V_g = \{v | v = [v_1, \ldots, v_n]^T \in R^n, v_i = 0 \text{ if } i \notin S_g\}.$$

The corresponding restriction matrices $R_b \in R^{n_1 \times n}$ and $R_g \in R^{n_2 \times n}$ are defined accordingly. We then consider the following nonlinear system

$$\mathcal{F}(Y) = R_b F(Y) + R_g (Y - X) = 0, \tag{3}$$

which we solve using the classical inexact Newton with backtracking (INB) algorithm. Denoting the solution by $\tilde{X}$, we define the solution operator G through the relation $\tilde{X} = G(X)$.

Algorithm 1 describes right-preconditioned inexact Newton algorithm with backtracking based on nonlinear elimination (INB-NE), as outlined in [24, 27].

Algorithm 1 Inexact Newton algorithm with backtracking based on nonlinear elimination (INB-NE)

Specify the initial guess $X^{(0)}$ and $k = 0$.
Initialize the partition: $S_b^{(0)}$ and $S_g^{(0)}$.
while $\|F(X^{(k)})\| > \epsilon_{\text{global-nonlinear-rtol}} \|F(X^{(0)})\|$ and $\|F(X^{(k)})\| > \epsilon_{\text{global-nonlinear-atol}}$ **do**

1. Compute a shifted starting point $\tilde{X}^{(k)} = G(X^{(k)})$.
2. Find the inexact Newton direction $d^{(k)}$ such that

$$F'(\tilde{X}^{(k)})d^{(k)} = -F(\tilde{X}^{(k)}). \tag{4}$$

3. Update $X^{(k+1)} = \tilde{X}^{(k)} + \lambda^{(k)} d^{(k)}$, where $\lambda^{(k)} \in (0, 1]$ is the damping parameter determined by a backtracking line search along $d^{(k)}$ such that

$$\|F(\tilde{X}^{(k)} + \lambda^{(k)} d^{(k)})\| \leqslant \theta \|F(\tilde{X}^{(k)})\|, \quad \theta \in (0, 1). \tag{5}$$

Set $k = k + 1$ and determine a new partition $S = S_b^{(k)} \bigcup S_g^{(k)}$.
end while

The most challenging aspect of INB-NE lies in identifying nonoverlapping sets of “bad” and “good” components, denoted as $S_b^{(k)}$ and $S_g^{(k)}$ $(k = 0, 1, ...)$. This partitioning is inherently problem-dependent and may vary across iterations. Crucially, the method’s computational efficiency is highly sensitive to the selection of “bad” components for elimination. An overly conservative strategy, in which all suspected components are sifted into $S_b^{(k)}$, expands the subspace problem’s dimension, increasing computational costs. Furthermore, the resulting larger subproblem may still suffer from slow convergence. On the other hand, insufficient removal of problematic components can fail to stabilize subsequent global Newton iterations, leading to lack of convergence.

Remark 1 The INB-NE algorithm does not ensure global convergence because the condition $\|F(X^{(k+1)})\| < \|F(\tilde{X}^{(k)})\|$ in (5) does not exclude the possibility of $\|F(X^{(k+1)})\| > \|F(X^{(k)})\|$ [22], potentially leading to divergence. However, this earliest right nonlinear preconditioner remains one of the most computationally attractive and commonly applied.

4 Elimination Strategies

In this work, we investigate three elimination strategies for the INB-NE algorithm applied to the backward-facing step flow problem, one based on pointwise residuals, one based on the degree nonlinearity in the governing equation, and a hybrid, specifically,

- **INB-NE-pointwise**: At the k-th iteration, the point-block residual vector at the mesh point $\mathbf{p}_i (0 \leqslant i \leqslant N-1)$ is denoted by

$$F_i^{\text{block}}(X^{(k)}) = \begin{bmatrix} F_{u,i}(X^{(k)}) \\ F_{v,i}(X^{(k)}) \\ F_{\omega,i}(X^{(k)}) \end{bmatrix}. \tag{6}$$

  The index set corresponding to the bad components are defined as

$$S_{\Omega_b}^{(k)} = \{i \mid \|F_i^{\text{block}}(X^{(k)})\|_\infty > \beta \|F(X^{(k)})\|_\infty,\ 0 \leqslant i \leqslant N-1\}, \tag{7}$$

  where β is a preselected constant. We simultaneously remove the (u, v, ω) triplet at nodal points $\mathbf{p}_i$ for all $i \in S_{\Omega_b}^{(k)}$, and the resulting subproblem is nonlinear due to the coupling of the three variables. The number of bad components to be eliminated depends sensitively on the parameter β, which has a significant impact on the effectiveness and efficiency of the preconditioner.

- **INB-NE-ω-field**: All vorticity ω components are designated as bad variables and remain fixed throughout the computation. This treatment renders the corresponding subsystem linear with respect to its own variable ω.

- **INB-NE-ω-field-uv**: We simultaneously eliminate the entire vorticity field while selectively removing specific nodal unknowns (u and v components) at points $\mathbf{p}_i$ for all $i \in S_{\Omega_b}^{(k)}$ as defined in equation (7).

To maintain computational efficiency, we restrict the size of the subproblem. An excessively large subproblem implies a dominant residual across much of the domain, leading to high computational costs. At the k-th step, the nonlinear preconditioners INB-NE-pointwise or INB-NE-ω-field-uv are applied only if the set $S_{\Omega_b}^{(k)}$ contains fewer than 15% of the total mesh points. Otherwise:

- For INB-NE-pointwise, the algorithm reverts to the standard Newton's method.
- For INB-NE-ω-field-uv, it defaults to the INB-NE-ω-field preconditioner.

5 Numerical Experiments

All numerical experiments are implemented using PETSc [2]. The global nonlinear problem is solved iteratively using the INB method, with convergence declared when the residual satisfies:

$$\|F(X^{(k)})\| \leqslant \max\{\epsilon_{\text{global-nonlinear-atol}}, \epsilon_{\text{global-nonlinear-rtol}} \|F(X^{(0)})\|\},$$

where $\epsilon_{\text{global-nonlinear-atol}}$ and $\epsilon_{\text{global-nonlinear-rtol}}$ denote the prescribed absolute and relative tolerances, respectively. Global Jacobian systems are solved by GMRES(50) with right overlapping restricted additive Schwarz (RAS) preconditioners [6], where each individual block is solved by the direct LU decomposition and the overlap is set to 1. The linear iterations terminate when

$$\|F(\tilde{X}^{(k)}) + J(\tilde{X}^{(k)})M_{RAS}^{-1}(M_{RAS}d^{(k)})\| \leqslant \epsilon_{\text{global-linear-rtol}}\|F(\tilde{X}^{(k)})\|,$$

where $\epsilon_{\text{global-linear-rtol}}$ controls the solution accuracy, and the RAS preconditioner is constructed as:

$$M_{RAS}^{-1} = \sum_{i=1}^{N} (R_i^0)^T J_i^{-1} R_i^\delta, \quad J_i = R_i^\delta J(\tilde{X}^{(k)})(R_i^\delta)^T.$$

In our numerical experiments, we set the initial guess to the zero vector. For nonlinear preconditioners, both global systems and subspace nonlinear problems are solved by INB with the parameters $\epsilon_{\text{global-nonlinear-rtol}} = 10^{-10}$, $\epsilon_{\text{global-nonlinear-atol}} = 10^{-12}$, $\epsilon_{\text{sub-nonlinear-rtol}} = 10^{-3}$. Note that a loose subproblem nonlinear tolerance keeps the number of Newton steps in the nonlinear preconditioning iterations low. We set the inexact stopping condition for global and subspace Jacobian systems as $\epsilon_{\text{global-linear-rtol}} = 10^{-6}$. For all of the tests, we set the initial partition as $S_b^{(0)} = \emptyset$ and $S_g^{(0)} = S$, i.e., we implement one-step of standard Newton iteration prior to applying nonlinear preconditioning.

5.1 Validation of the discretization scheme against benchmarks

Before describing the convergence behavior of the solvers, first validate the finite difference discretization scheme through comparison of velocity and vorticity profiles for backward-facing step flow against established benchmark results. At the Reynolds number $Re = 800$, Fig. 1 presents a systematic comparison between our numerical solutions and the reference data from Gartling [11]. The results demonstrate excellent convergence behavior as the mesh resolution is progressively refined from 1201×41 to 1801×61 and finally to 2401×81. Both velocity and vorticity profiles show strong agreement with the published benchmark solutions in [11], confirming the accuracy and reliability of our numerical approach. Note that the magnitude of the v velocity component (transverse to the channel flow) is typically two orders of magnitude lower than longitudinal, so its relative error is somewhat higher. (We note from the literature on this configuration, which has been well studied both computationally and experimentally, that essentially two-dimensional laminar flow can be achieved up to Reynolds numbers of about 800. Steady-state 2D laminar numerical solutions exist above this, so we push Reynolds higher, to 1000, in order to stress Newton's method.)

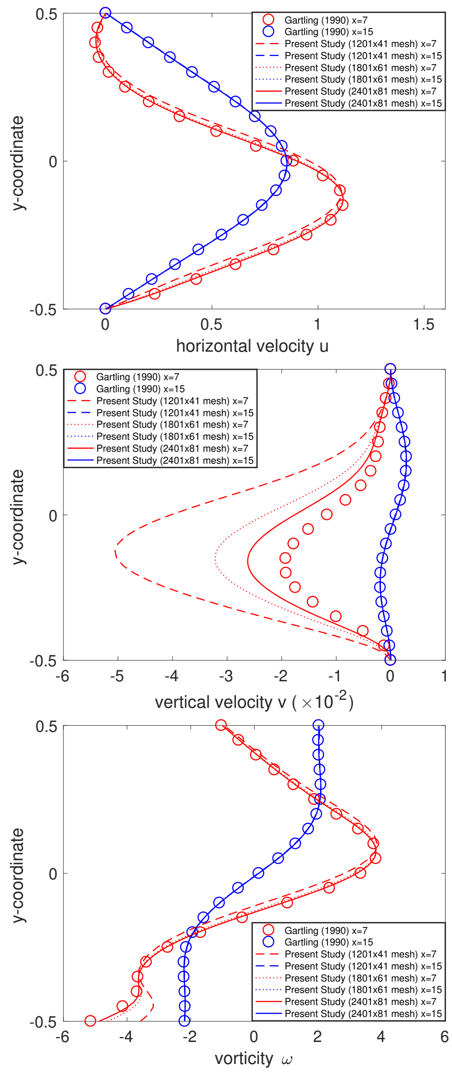

Fig. 1: Comparison of u, v, ω profiles at various downstream locations for $Re = 800$. (Note the zoomed-in scale of the vertical velocity.)

5.2 Comparison of INB and INB-NE with different elimination strategies

We set the parameter $\beta = 0.05$ in (7) for both INB-NE-pointwise and INB-NE-ω-field-uv methods. Fig. 2 presents a comparison of the convergence history of the Newton residuals using four different methods: INB, INB-NE-pointwise, INB-NE-

ω-field, and INB-NE-ω-field-uv, on a 1201×41 mesh at various Reynolds numbers. The standard INB method fails to converge for Reynolds numbers exceeding approximately 400, as evidenced by the prolonged plateau in the nonlinear residuals. Notably, the three INB-NE variants exhibit oscillatory nonlinear residuals at certain stages, which can be attributed to the monotonicity test being performed from the shifted point $\tilde{X}^{(k)}$ rather than $X^{(k)}$. Among these, INB-NE-pointwise shows particularly unstable behavior, with residuals eventually diverging and failing to achieve convergence when $Re = 800$. In contrast, INB-NE-ω-field and INB-NE-ω-field-uv successfully converge across all tested cases. However, INB-NE-ω-field-uv demonstrates superior efficiency, requiring significantly fewer nonlinear iterations at $Re = 800$ compared to INB-NE-ω-field.

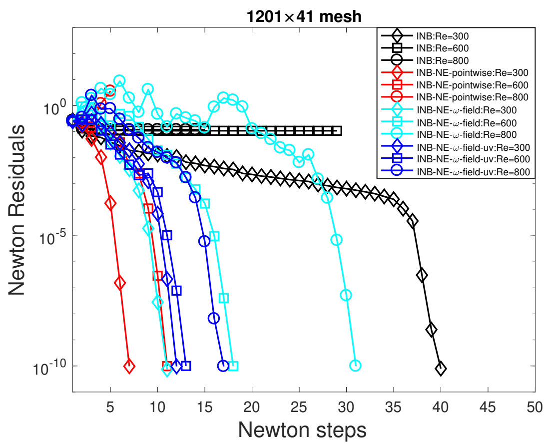

Fig. 2: The convergence history of the Newton residuals for the backward-facing step flow using INB and INB-NE with three different nonlinear elimination strategies on the 1201×41 mesh, respectively.

5.3 Impact of the parameters in nonlinear elimination

The INB-NE-ω-field algorithm selects the entire ω-field as the set of bad variables, which remains fixed throughout the computation. In contrast, for both INB-NE-pointwise and INB-NE-ω-field-uv, the set of bad variables dynamically changes based on the parameter β, which plays a crucial role in their global convergence performance.

To examine the parametric influence, Table 1 presents the number of Newton iterations and the execution time required by INB-NE-pointwise, INB-NE-ω-field and INB-NE-ω-field-uv for different values of β across Reynolds numbers ranging

Table 1: A comparison of the number of Newton iterations and the execution time of INB-NE-pointwise, INB-NE-ω-field and INB-NE-ω-field-uv corresponding to β values at different Reynolds numbers ."-" indicates that the case fails to converge. The mesh size is 1201×41 and 10 (10×1) processors are used.

Re	INB-NE-ω-field		β	INB-NE-pointwise		NE-ω-field-uv	
	Iter	Time(s)		Iter	Time(s)	Iter	Time(s)
400	14	30.72	0.01	-		10	33.89
			0.02	-		11	37.11
			0.05	8	18.02	15	51.50
			0.07	8	19.46	15	51.30
			0.1	10	25.21	12	40.89
			0.2	-		12	39.04
			0.5	-		12	34.06
600	18	48.82	0.01	-		13	54.06
			0.02	-		14	53.99
			0.05	11	31.93	13	55.64
			0.07	11	32.19	15	55.11
			0.1	-		16	57.06
			0.2	-		22	74.42
			0.5	-		21	61.17
800	31	107.68	0.01	-		17	66.45
			0.02	-		19	87.46
			0.05	-		17	77.74
			0.07	-		30	128.46
			0.1	-		22	94.89
			0.2	-		21	89.52
			0.5	-		25	91.59
1000	-		0.01	-		40	233.45
			0.02	-		53	316.26
			0.05	-		54	287.51
			0.07	-		-	
			0.1	-		33	162.60
			0.2	-		51	214.09
			0.5	-		24	102.87

from 400 to 1000, using a 1201×41 mesh on 10 (10×1) processors. (Unpreconditioned INB is not listed due to the lack of convergence noted in Fig. 2.) The experiments show that INB-NE-pointwise exhibits significant sensitivity to β selection, with convergence failures observed at certain parameter combinations. This sensitivity becomes more pronounced with increasing Reynolds numbers, requiring progressively stricter β selection for guaranteed convergence. In contrast, the INB-NE-ω-field scheme demonstrates robust convergence behavior across all tested cases except at $Re = 1000$. Most notably, the INB-NE-ω-field-uv variant maintains superior parametric flexibility, achieving successful convergence even at $Re = 1000$ through appropriate β selection.

As shown in Table 1, while INB-NE-pointwise offers has the best efficiency when convergent and INB-NE-ω-field offers the next smaller runtimes at lower Reynolds numbers, INB-NE-ω-field-uv provides the best balance between reliability and performance, and outlasts the other preconditioners in convergence at high Reynolds numbers ($Re \geqslant 800$).

6 Conclusions

We demonstrate that nonlinear elimination techniques can significantly enhance the performance of inexact Newton methods for solving the challenging steady backward-facing step flow problem when started from a "cold" initial iterate. Experiments confirm that the "best of both worlds" hybrid method is more reliable than both pure pointwise and full field-based elimination approaches. This provides insights for developing effective nonlinear solvers in other CFD applications. Future research could explore automated parameter selection and extension to three-dimensional problems.

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