Overlapping Subspaces and Singular Systems with Application to Isogeometric Analysis

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1 Introduction

In this paper, we propose a novel framework for numerical methods for partial differential equations (PDEs), inspired by isogeometric analysis (IGA) based on local tensor-product splines that allow local refinement. This leads to a potentially singular linear system, which is handled by a Krylov linear solver. The framework may offer computational advantages in dealing with spaces like Hierarchical B-splines, T-splines, and LR-splines.

Consider the problem: find $u \in V$ such that

$$Au = f, \tag{1}$$

where $A : V \to V^*$ is a linear operator and $f \in V^*$, with V being a Hilbert space. Assuming V is finite-dimensional, one usually rewrite (1) in matrix form by choosing a basis Φ for V. Instead, we take a different approach, further assuming that V is given as the sum, in general not direct, of closed subspaces $V_i \subset V$, $i = 1, \ldots, n$,

$$V = \sum_{i=1}^{n} V_i, \tag{2}$$

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and that we have a basis Φ_i for each subspace V_i. We then consider the disjoint union

$$\underline{\Phi} = \coprod_{i=1}^{n} \Phi_i, \tag{3}$$

which generally does not form a basis for V, as its elements may be linearly dependent (or even duplicated). We represent the solution u as a linear combination of elements of $\underline{\Phi}$ and, using $\underline{\Phi}$ instead of Φ, rewrite (1) as a linear system, which may be singular. Therefore, such a linear system cannot be solved by standard direct methods but can be handled by iterative methods. In particular, it is known that Krylov methods perform well with singular systems, provided that the right-hand side lies in the range of the system matrix, see e.g. [11, 10, 12]. At the same time, preconditioned Krylov methods are currently the most efficient solvers when a good preconditioner is available. Following the domain decomposition approaches, and in particular the subspace correction paradigm, preconditioners can be constructed from the approximate solvers of the restrictions of A to each V_i. Our strategy is to carefully select the V_i, for example, we could consider V_i with a tensorial structure, that can be exploited for (possibly inexact) fast solvers.

The framework outlined above is unconventional but not unprecedented in the literature. Griebel's 1994 paper [7] in the context of finite elements proposed abandoning the basis in favor of a generating system as in (3) to simplify preconditioner construction. Subsequent research has utilized generating systems in wavelet methods and sparse grid contexts.

We think this framework is potentially very beneficial for isogeometric analysis (IGA). IGA, introduced in 2005 [9, 3], aims to bridge the gap between numerical simulations and Computer-Aided Design (CAD). Specifically, IGA follows the isoparametric paradigm and adopts B-splines (or NURBS and other spline extensions) as basis functions. This allows the unknown fields in numerical simulations to be represented by splines, just as splines are used to parameterize geometry in CAD. In both IGA and CAD, each patch is typically derived from a tensor-product basis. However, spline extensions for mesh refinement such as Hierarchical B-splines [6, 2], T-splines [13, 4], and LR-splines [5, 1], feature a local tensor-product structure. These spaces may be seen as unions of tensor-product subspaces V_i, but often identifying a global basis Φ is challenging, while working with subspace bases Φ_i and their disjoint union $\underline{\Phi}$ is a significant simplification. Furthermore, using $\underline{\Phi}$ for representing the unknown, we may retain the computational advantages related to the tensor-product structure of the V_i, e.g., the possibility of fast local solvers.

The structure of this work is the following: Section 2 sets the notation, Sections 3–4 reframe Krylov methods and subspace correction preconditioners in the proposed setting, Section 5 contains a numerical benchmark in the framework of IGA and Section 6 contains our conclusions.

2 Abstract setting

We assume that V is a finite dimensional vector space with scalar product $(\cdot,\cdot)_V$ and norm $||\cdot||_V$. Given the spaces V_i, for $i = 1,\ldots,n$, and V, as in (2), we introduce the extended space

$$\underline{V} := \prod_{i=1}^{n} V_i, \tag{4}$$

endowed with the norm $||\underline{v}||_{\underline{V}} := \left(\sum_{i=1}^{n} ||v_i||_V^2\right)^{1/2}$, for all $\underline{v} = (v_1,\ldots,v_n) \in \underline{V}$, and reformulate the abstract problem (1) on $\underline{V}$, which will lead to a linear system for the representation of the solution with respect to $\underline{\Phi}$ defined in (3), the basis of $\underline{V}$, instead of Φ, the basis of V. For this purpose, each $\underline{v} \in \underline{V}$ is associated to a $v \in V$ via the following sum operator $S : \underline{V} \to V$

$$S\underline{v} = S(v_1,\ldots,v_n) = \sum_{i=1}^{n} v_i \in V \tag{5}$$

Its adjoint $S^* : V^* \to \underline{V}^* = \prod_i V_i^*$ is then injective and associates to f the tuple of the restrictions of f to the V_i, that is $S^* f = (f|_{V_1},\ldots,f|_{V_n})$. The *extended problem* is the pull-back of problem (1) on $\underline{V}$, and reads

$$\underline{Au} = \underline{f}, \tag{6}$$

where $\underline{f} := S^* f$ and $\underline{A} : \underline{V} \to \underline{V}^*$ is defined by

$$\underline{A} := S^* A S \tag{7}$$

that is

$$\underline{A}(u_1,\ldots,u_n) = (w|_{V_1},\ldots,w|_{V_n}), \text{ with } w = \sum_{i=1}^{n} A u_i. \tag{8}$$

The *extended problem* is equivalent to problem (1) as stated in the next result.

Theorem 1 *If $A : V \to V^*$ is an isomorphism, then*

$$\ker \underline{A} = \ker S, \tag{9}$$

and

$$Au = f \textit{ if and only if } u = S\{\underline{u} \in \underline{V} : \underline{Au} = \underline{f}\}. \tag{10}$$

Proof. Recalling that S is surjective and S^* is injective, (9) follows. Thanks to S^* being injective and A an isomorphism, we also have

$$\underline{Au} = \underline{f} \Leftrightarrow S^*(AS\underline{u}) = S^* f \Leftrightarrow AS\underline{u} = f \Leftrightarrow u = S\underline{u},$$

which is (10). □

3 Krylov methods

In this section we revise and adapt to our framework the theory of preconditioned Krylov methods, with emphasis on the Conjugate Gradient (CG) method, following [8]. In particular, we relate Krylov methods for the extended problem (6) to Krylov method for the original problem(1).

Assume that a left preconditioner $\underline{B} : \underline{V}^* \to \underline{V}$ for the extended problem (6) is given. We associate to it a left preconditioner $B : V^* \to V$ for the original problem (1) as follows:

$$B = S\underline{B}S^*; \tag{11}$$

then, the following commuting diagram holds

$$\begin{tikzcd}
\underline{V} \arrow[r, "\underline{A}"] \arrow[d, "S"] & \underline{V}^* \arrow[r, "\underline{B}"] & \underline{V} \arrow[d, "S"] \\
V \arrow[r, "A"] & V^* \arrow[u, "S^*"] \arrow[r, "B"] & V
\end{tikzcd} \tag{12}$$

The original and extended (left) preconditioned operators are $T := BA$ and $\underline{T} := \underline{BA}$, respectively, such that

$$TS = S\underline{T}. \tag{13}$$

Finally, the (left) preconditioned extended problem reads

$$\underline{Tu} = \underline{Bf}. \tag{14}$$

Being $\underline{T} : \underline{V} \to \underline{V}$ an endomorphism on $\underline{V}$ (though possibly singular) we can employ Krylov methods for solving (14). Denoting $\underline{y} = \underline{Bf}$, Krylov methods provide at each step s an approximate solution $\underline{u}_s$ in the Krylov subspace

$$K_s(\underline{T}, \underline{y}) := \operatorname{span}\{\underline{y}, \underline{Ty}, \ldots, \underline{T}^{s-1}\underline{y}\}, \tag{15}$$

such that a suitable optimality condition is satisfied.

The convergence of Krylov methods in the singular case is not guaranteed in general: it may happen that the solution $\underline{x}$ of $\underline{Tx} = \underline{y}$ does not belong to any Krylov space $K_s(\underline{T}, \underline{y})$. Fortunately, this does not happen in our case under reasonable conditions, as stated in the following proposition.

Proposition 1 *Under the assumption that T is an isomorphisms, for all $f \in V$ the system* (14) *admits a Krylov solution.*

Proof. The kernel of a composition always contains the kernel of the first map and so by definition

$$\ker \underline{T} = \ker(\underline{B}S^*AS) \supseteq \ker S.$$

Similarly, since T is an isomorphism and using (13) we have

$$\ker S = \ker TS = \ker S\underline{T} \supseteq \ker \underline{T}.$$

Therefore

$$\ker \underline{T} = \ker S. \tag{16}$$

From (16), then (13), using that T is an isomorphism, and finally (16) again we have

$$\ker \underline{T}^2 = \ker S\underline{T} = \ker TS = \ker S = \ker \underline{T},$$

i.e., $\ker \underline{T}^2 = \ker \underline{T}$. This guarantees that [10, Theorem 2] $\underline{T}\underline{x} = \underline{z}$ admits a Krylov solution for any $\underline{z} \in \mathscr{R}(\underline{T})$. □

The assumption of Theorem 1 is natural in our framework. In particular, A is taken non-singular by assumption, $\underline{B}$ is often coercive by construction

$$\forall \underline{g} \in \underline{V}^*, \quad \underline{g} \neq 0 \Rightarrow \langle \underline{B}\underline{g}, \underline{g} \rangle > 0;$$

then B is also coercive (since $\langle Bg, g \rangle = \langle \underline{B}S^* g, S^* g \rangle$, and S^* injective) and thus B is an isomorphism, which makes T an isomorphism as well.

Theorem 1 assumes exact arithmetic and the exact calculation of the matrix and right hand side of the preconditioned extended problem (14). However, numerical errors typically occur and, when $\underline{T}$ is singular, they may lead to the unsolvability of the perturbed system. Assume, for example, that the left-hand side of $\underline{T}$ is exactly computed (up to machine precision) and the right-hand-side is affected by a quadrature error $\underline{e}$, that is the system reads $\underline{A}\underline{u} = \underline{f} + \underline{e}$. Let's further decompose $\underline{e}$ as $\underline{e} = \underline{e}_{\mathscr{R}} + \underline{e}_{\mathscr{K}}$, with $\underline{e}_{\mathscr{R}}$ in the range of S^*. The effect of $\underline{e}_{\mathscr{R}}$ can be analyzed by means of standard methods such as the Strang lemma, while $\underline{e}_{\mathscr{K}}$, which is outside the range of S^* and then of $\underline{A}$, prevents the existence of the solution of the perturbed problem. In this situation, it is required that the stopping criterion of the Krylov iteration is set with a threshold on the residual norm that cannot be smaller than the norm of $\underline{e}_{\mathscr{K}}$.

3.1 CG and MINRES

If A and $\underline{B}$ are self-adjoint and coercive

$$\langle Av, w \rangle = \langle Aw, v \rangle, \qquad \langle Av, v \rangle > 0, \qquad \forall v, w \in V, \tag{17}$$

$$\langle \underline{B}\underline{v}, \underline{w} \rangle = \langle \underline{B}\underline{w}, \underline{v} \rangle, \qquad \langle \underline{B}\underline{v}, \underline{v} \rangle > 0, \qquad \forall \underline{v}, \underline{w} \in \underline{V}^*. \tag{18}$$

then also $\underline{A}$ is self-adjoint and B is both self-adjoint and coercive

$$\langle \underline{A}\underline{v}, \underline{w} \rangle = \langle \underline{A}\underline{w}, \underline{v} \rangle, \qquad \forall \underline{v}, \underline{w} \in \underline{V}, \tag{19}$$

$$\langle Bv, w \rangle = \langle Bw, v \rangle, \qquad \langle Bv, v \rangle > 0, \qquad \forall v, w \in V^*. \tag{20}$$

We can indeed apply CG, for which the convergence histories of the original and extended problem coincide.

Theorem 2 *Under the assumptions above, let u_s and $\underline{u}_s$ be the iterates at step s of the (preconditioned) CG method applied to the original problem* (1) *and the extended problem* (6)*, respectively. Then for all s,*

$$S\underline{u}_s = u_s \tag{21}$$

$$||\underline{e}_s||_{\underline{A}} = ||e_s||_A, \tag{22}$$

where $\underline{e}_s \in \underline{V}$ is the error at step s of the CG method applied to the extended problem, i.e., the difference between a chosen solution $\underline{u}$ and $\underline{u}_s$, while $e_s \in V$ is the corresponding error obtained by the CG method applied to the original problem, and $||\cdot||_A = \langle A\cdot,\cdot\rangle$ and $||\cdot||_{\underline{A}} = \langle \underline{A}\cdot,\cdot\rangle$ denote the norms or seminorms associated to A and $\underline{A}$, respectively.

Proof. For any positive integer s, we have by definition of T and $\underline{T}$ that

$$T^s S = S\underline{T}^s,$$

and then

$$T^s Bf = S\underline{T}^s \underline{Bf},$$

yielding the correspondence of the Krylov spaces generated in V and $\underline{V}$:

$$K_s(T, Bf) = SK_s(\underline{T}, \underline{Bf}). \tag{23}$$

By construction any CG solution of (1), respectively,(6) fulfill the following optimality condition

$$u_s = \operatorname*{argmin}_{v\in K_s(T,Bf)} ||u-v||_A, \quad \text{and} \quad \underline{u}_s \in \operatorname*{argmin}_{\underline{v}\in K_s(\underline{T},\underline{Bf})} ||\underline{u}-\underline{v}||_{\underline{A}}$$

By (23) and

$$||\underline{v}||^2_{\underline{A}} = \langle \underline{Av}, \underline{v}\rangle = \langle AS\underline{v}, S\underline{v}\rangle = ||S\underline{v}||^2_A.$$

we have $S\underline{u}_s = u_s$, i.e., (21) and (22).

□

In a similar way, but without the need of coercivity of $\langle A\cdot,\cdot\rangle$, i.e., assuming only that A is a self-adjoint isomorphism satisfying

$$\langle Av, w\rangle = \langle Aw, v\rangle, \qquad \forall v, w \in V, \tag{24}$$

we can still apply MINRES and have the following equivalence.

Theorem 3 *Under the assumptions* (24), (18)*, let u_s and $\underline{u}_s$ be the iterates at step s of the (preconditioned) MINRES method applied to the original problem* (1) *and the* *extended problem* (6)*, respectively. Then for all s,*

$$S\underline{u}_s = u_s \tag{25}$$

and

$$||\underline{r}_s||_{\underline{B}} = ||r_s||_B, \tag{26}$$

where $\underline{r}_s \in \underline{V}^*$ *is the residual at step s of the MINRES method applied to the extended problem,* $r_s \in V^*$ *is the residual of the MINRES method applied to the original problem and* $||\cdot||_B = \langle B\cdot,\cdot\rangle$ *and* $||\cdot||_{\underline{B}} = \langle \underline{B}\cdot,\cdot\rangle$ *denote the norms associated to B and* $\underline{B}$*, respectively.*

Proof. As shown in (23) the Krylov subspaces correspond by S. Similarly to the CG, any MINRES solution of (1), respectively,(6) fulfill the following optimality conditions

$$u_s = \operatorname*{argmin}_{v\in K_s(T,Bf)} ||Av - f||_B, \quad \text{and} \quad \underline{u}_s \in \operatorname*{argmin}_{\underline{v}\in K_s(\underline{T},\underline{B}\underline{f})} ||\underline{A}\underline{v} - \underline{f}||_{\underline{B}}.$$

By (23), $||S^*v||_{\underline{B}} = ||v||_B$ and

$$\underline{A}\underline{v} - \underline{f} = S^*(AS\underline{v} - f)$$

we have $S\underline{u}_s = u_s$, i.e., this proves (25) and (26). □

3.2 GMRES

The standard GMRES method minimizes at each step the euclidean norm of the residual. It follows that the iterate are basis dependent and differ between the original and the extended problem. At the same time, the behavior of GMRES mainly depends on the spectrum of the preconditioned problem, which is the same for the two problems, original and extended, up to the possible kernel of the latter.

Theorem 4 *Under the assumption that T is an isomorphism, the spectrum* $\sigma(\underline{T})$ *of* $\underline{T}$*, and the spectrum* $\sigma(T)$ *of T are related by*

$$\sigma(\underline{T}) \setminus \{0\} = \sigma(T).$$

Proof. For $\lambda \in \sigma(\underline{T}) \setminus \{0\}$ there exists $\underline{v}_\lambda \notin \ker S$ such that $\underline{T}\underline{v}_\lambda = \lambda\underline{v}_\lambda$. From (13)

$$TS\underline{v}_\lambda = S\underline{T}\underline{v}_\lambda = \lambda S\underline{v}_\lambda$$

and we see that $S\underline{v}_\lambda$ is an eigenvector for T and $\lambda \in \sigma(T)$. Thus $\sigma(\underline{T}) \setminus \{0\} \subseteq \sigma(T)$.

Similarly if $\lambda \in \sigma(T)$ there exists $w_\lambda \neq 0$ such that $\lambda w_\lambda = Tw_\lambda$. Let

$$\underline{w}_\lambda = \lambda^{-1}\underline{B}S^*Aw_\lambda, \tag{27}$$

then we have

$$S\underline{w}_\lambda = \lambda^{-1} S\underline{B}S^* A w_\lambda = \lambda^{-1} T w_\lambda = w_\lambda \tag{28}$$

and, using (28) and then (27), $\underline{T}\underline{w}_\lambda = \underline{B}S^* A S \underline{w}_\lambda = \underline{B}S^* A w_\lambda = \lambda \underline{w}_\lambda$, and therefore $\lambda \in \sigma(\underline{T})$. □

4 Subspace correction methods

We briefly review, in this section, the so-called *successive subspace correction* (SSC) methods, see e.g., [14], and frame them into our setting. We assume local preconditioners B_i, for the restriction of A to the subspaces V_i, are given, denote their extensions to the whole $\underline{V}$ as $\underline{B}_i$, that is

$$\underline{B}_i := \begin{pmatrix} 0 & \dots & 0 \\ \vdots & B_i & \vdots \\ 0 & \dots & 0 \end{pmatrix} \tag{29}$$

and define the local preconditioned operators $T_i := S\underline{B}_i S^* A$, and $\underline{T}_i := \underline{B}_i \underline{A}$. Consider then the inexact block-Gauss-Seidel preconditioner $\underline{B}_{\text{ssc}}$, whose application $\underline{w} = \underline{B}_{\text{ssc}} \underline{v}$ is given by the following algorithm:

$\underline{w} \leftarrow 0$
for $j = 1, \dots, r$ **do**
  $\underline{t} \leftarrow \underline{B}_{i_j} \underline{v}$
  $\underline{w} \leftarrow \underline{w} + \underline{t}$
  $\underline{v} \leftarrow \underline{v} - \underline{A}\underline{t}$
end for

where $(i_1, \dots, i_r)$ defines the sequence of subspace corrections. In closed form they can be written as

$$\underline{B}_{\text{ssc}} := \sum_{j=1}^{r} \underline{B}_{i_j} \prod_{\ell=1}^{j-1} (\underline{\mathbb{1}} - \underline{A}\underline{B}_{i_\ell}). \tag{30}$$

The preconditoned operator now reads then

$$\underline{T}_{\text{ssc}} := \underline{B}_{\text{ssc}} \underline{A} = \sum_{j=1}^{r} \underline{T}_{i_j} \prod_{\ell=1}^{j-1} (\underline{\mathbb{1}} - \underline{T}_{i_\ell}).$$

In this case $B_{\text{ssc}} = S\underline{B}_{\text{ssc}} S^*$ is a multiplicative Schwarz preconditioner (or inexact solver) for A.

Examples of $\underline{B}_{\text{ssc}}$ are the *forward* SSC $\underline{B}_{\text{ssc}}^{\text{forw}}$, for $r = n$ and $i_j = j$, and the *backward* SSC $\underline{B}_{\text{ssc}}^{\text{back}}$ to $r = n$ and $i_j = n + 1 - j$.

A symmetric preconditioner can also be constructed by a suitable composition of $\underline{B}_{\text{ssc}}^{\text{forw}}$, $\underline{B}_{\text{ssc}}^{\text{back}}$ and the diagonal blocks of $\underline{A}$.

5 Numerical examples

We consider the solution of the (variational) Poisson problem:

$$\text{find } u \in V \subset H^1_0(\Omega) \quad \text{s.t.} \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \quad \forall v \in V\,,$$

where Ω is the image of the parametric domain

$$\widehat{\Omega} := [1,2] \times [\frac{\pi}{4}, \frac{3\pi}{4}]$$

under the polar mapping

$$F(\rho,\theta) = \big(\rho\cos(\theta), \rho\sin(\theta)\big)^T,$$

and f is a piecewise-constant function (with random constant coefficients over a uniform 4^2 tessellation of the parametric domain). The test and trial space V is defined as $V = V_0 + \sum_{k=1}^{n} \big(V_k^x + V_k^y\big)$, where:

- V_0 is the push forward through F of the spline space of degree p and continuity $p-1$, built over $\widehat{\Omega}$ with uniform knots vectors of size $\frac{1}{2^{L+2}}(1, \frac{\pi}{2})$ (with $L \geq 0$) and satisfying homogeneous Dirichlet boundary conditions. The support of this space is depicted in Fig.1;
- V_k^x is the push forward through F of the spline space of degree p and continuity $p-1$, built over $\widehat{\Omega}_k^x = [1, \frac{3}{2}] \times [\frac{\pi}{4}, \frac{\pi}{4}\big(1 + \frac{1}{2^{k-1}}\big)]$, with uniform knots vectors of size $\frac{1}{2^{L+2}}(1, \frac{\pi}{2^{k+1}})$ (with $L \geq 0$) and satisfying homogeneous Dirichlet boundary conditions.The supports of the spaces V_1^x and V_2^x (with $L = 0$) are depicted in Fig. 2a and 2b, respectively;
- V_k^y is the push forward through F of the spline space of degree p and continuity $p-1$, built over $\widehat{\Omega}_k^y = [1, 1 + \frac{1}{2^k}] \times [\frac{\pi}{4}, \frac{5\pi}{8}]$, with uniform knots vectors of size $\frac{1}{2^{L+2}}(\frac{1}{2^k}, \frac{\pi}{2})$ (with $L \geq 0$) and satisfying homogeneous Dirichlet boundary conditions. The support of the spaces V_1^y and V_2^y (with $L = 0$) are depicted in Fig. 2c and 2d, respectively.

The iterative linear solver is a symmetric Gauss-Seidel preconditioned Conjugate Gradient, stopped when the discrete residual is $< 10^{-11}$ or the number of iterations reached the upper limit of 1000. We tested the proposed method with $n = 2$ (corresponding to 5 overlapping domains) with different values of the mesh parameter L for degrees $p = 2$ (Table 1) and $p = 3$ (Table 2), where we observe the effectiveness of the preconditioned solver: in both cases $p = 2$ and $p = 3$ the number of iterations needed for the convergence of the preconditioned approach increases for the coarser meshes (i.e., low values of L) and then remain almost constant for the finer meshes, as expected.

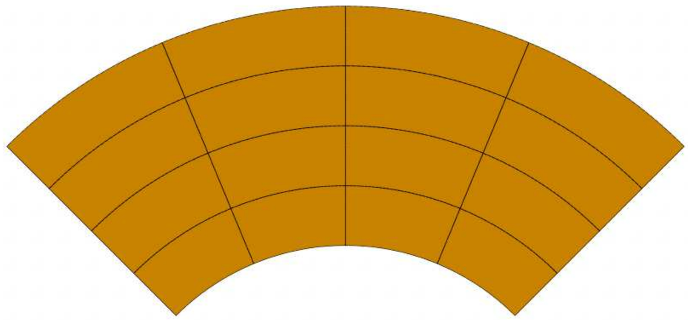
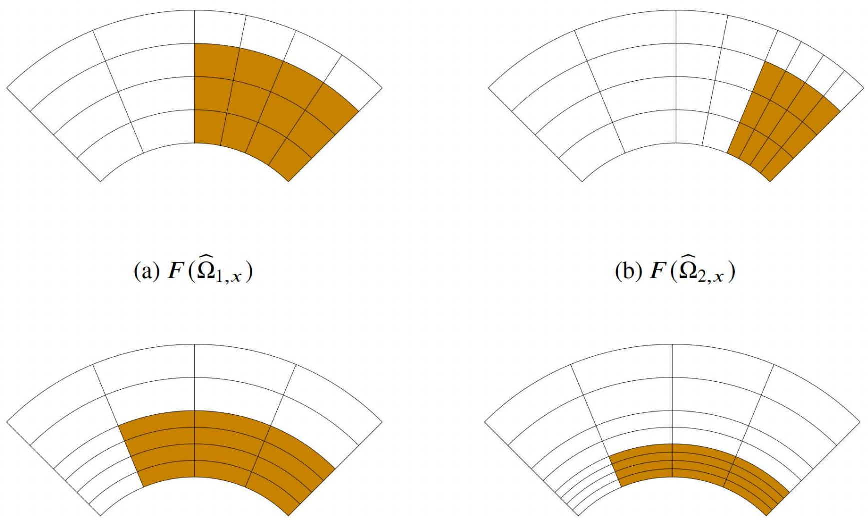


Fig. 1: 2D case: the physical domain $F(\widehat{\Omega}_0)$ associated to the space V_0 for $L = 0$.



(a) $F(\widehat{\Omega}_{1,x})$

(b) $F(\widehat{\Omega}_{2,x})$

(c) $F(\widehat{\Omega}_{1,y})$

(d) $F(\widehat{\Omega}_{2,y})$

Fig. 2: 2D case: the different physical domains associated to the spaces V_1^x (2a), V_2^x (2b), V_1^y (2c) and V_2^y (2d) for $L = 0$.

		CG		PCG-GS	
L	DOFs	N. iters	Residual	N. iters	Residual
0	40	36	$3.15 \cdot 10^{-14}$	20	$2.17 \cdot 10^{-12}$
1	204	73	$7.60 \cdot 10^{-12}$	26	$5.01 \cdot 10^{-12}$
2	916	105	$9.76 \cdot 10^{-12}$	28	$1.43 \cdot 10^{-11}$
3	3,876	167	$9.32 \cdot 10^{-12}$	31	$2.60 \cdot 10^{-11}$
4	15,940	338	$8.38 \cdot 10^{-12}$	34	$3.61 \cdot 10^{-11}$
5	64,644	685	$9.86 \cdot 10^{-12}$	35	$6.51 \cdot 10^{-11}$
6	260,356	1,000	$4.33 \cdot 10^{-7}$	35	$1.25 \cdot 10^{-10}$
7	1,044,996	1,000	$2.47 \cdot 10^{-2}$	35	$1.89 \cdot 10^{-10}$

Table 1: Number of iterations w.r.t. the mesh parameter L, for the case with degree $p = 2$ and $n = 2$ (i.e., 5 overlapping domains).

		CG		PCG-GS	
L	DOFs	N. iters	Residual	N. iters	Residual
0	49	54	$2.67 \cdot 10^{-12}$	29	$1.26 \cdot 10^{-12}$
1	221	167	$5.82 \cdot 10^{-12}$	45	$4.97 \cdot 10^{-12}$
2	949	253	$8.74 \cdot 10^{-12}$	48	$1.27 \cdot 10^{-11}$
3	3,941	312	$9.89 \cdot 10^{-12}$	55	$2.63 \cdot 10^{-11}$
4	16,069	400	$9.38 \cdot 10^{-12}$	58	$5.09 \cdot 10^{-11}$
5	64,901	639	$9.97 \cdot 10^{-12}$	60	$7.22 \cdot 10^{-11}$
6	260,869	1,000	$1.01 \cdot 10^{-7}$	59	$1.60 \cdot 10^{-10}$
7	1,046,021	1,000	$1.45 \cdot 10^{-2}$	58	$2.61 \cdot 10^{-10}$

Table 2: Number of iterations w.r.t. the mesh parameter L, for the case with degree $p = 3$ and $n = 2$ (i.e., 5 overlapping domains).

6 Conclusions

In this work, we proposed a framework based on well-known tools such as Krylov solvers and preconditioners based on subspace corrections. The novelty we introduce is working with a linear system obtained from the disjoint union of the degrees of freedom of the subspaces. This approach has the advantage of preserving the original structure of the subspaces, for example, the tensor structure in the case of isogeometric discretizations, possibly allowing efficient (exact or inexact) solvers on the subspaces. We believe this is a promising direction, although in this work it is only sketched out and deserves further investigation and development.

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