

# BDDC Preconditioners for 3D Virtual Element Discretizations of Saddle-Point Problems

Simone Scacchi<sup>[0000-0001-6011-784X]</sup>,  
Tommaso Bevilacqua<sup>[0000-0002-0828-5018]</sup>,  
Franco Dassi<sup>[0000-0001-5590-3651]</sup>, and  
Stefano Zampini<sup>[0000-0002-0435-0433]</sup>

## 1 Introduction

The Virtual Element Method (VEM) is a recent technology for the numerical approximation of Partial Differential Equations (PDEs) [6]. Its key feature is to provide a generalization of the Finite Element Method to deal with computational grids constituted by elements of arbitrary polygonal or polyhedral shape. Effective VEM discretizations have been developed for several PDEs; the interested reader should consult the recent special issue [5] and the book [1] for further details.

Like other numerical techniques for PDEs, when the mesh size is refined or the order of approximation is increased, the condition number of the linear system arising from the discretization process increases. Therefore, the development of effective linear solvers is crucial to reduce the computational costs. Some recent studies have proposed multigrid and domain decomposition preconditioners for scalar elliptic equations in primal form: see [2, 3] for multigrid preconditioners, [8, 9, 15, 16] for Balancing Domain Decomposition by Constraints (BDDC) and Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) preconditioners, and [11, 12] for Overlapping Additive Schwarz preconditioners.

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Simone Scacchi

Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy  
e-mail: simone.scacchi@unimi.it

Tommaso Bevilacqua

Department Mathematik/Informatik, Universität zu Köln  
Weyertal 86-90, 50931 Köln, Germany e-mail: tommaso.bevilacqua@uni-koeln.de

Franco Dassi

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, Via Roberto  
Cozzi 55 - 20125 Milano, Italy e-mail: franco.dassi@unimib.it

Stefano Zampini

Extreme Computing Research Center, King Abdullah University of Science and Technology,  
Thuwal 23955, Saudi Arabia e-mail: stefano.zampini@kaust.edu.sa

The aim of this work is to review Balancing Domain Decomposition by Constraints (BDDC) preconditioners for virtual element discretizations of three-dimensional scalar elliptic equations in mixed form and Stokes equations, as presented in [14] and [10]. The Virtual Element Method (VEM) is a numerical technology for the solution of partial differential equations on computational grids constituted by polygonal or polyhedral elements of very general shape. The proposed BDDC preconditioners allow to use conjugate gradient iterations, albeit the algebraic linear systems arising from the discretization of the problems are indefinite and saddle point. The condition number of the resulting positive definite preconditioned linear system is adaptively controlled by means of deluxe scaling operators and suitable local generalized eigenvalue problems for the selection of optimal primal constraints. Numerical results confirm the scalability of the resulting parallel solver and the reliability of the adaptive procedure.

## 2 Saddle-point problems

We consider two different PDE problems which require the solution of a saddle-point linear system, i.e., the stiffness matrix can be written as

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times n}$ , then the whole linear system  $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ .

In a very general setting the variational formulation of such problems is the following. Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ , whose boundary is denoted by  $\partial\Omega$ . We look for a couple of functions  $(\mathbf{u}, q) \in \mathbf{V} \times Q$  which satisfy

$$\begin{cases} \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) = c_1(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \\ \mathbf{b}(\mathbf{u}, q) = c_2(q) \quad \forall q \in Q, \end{cases} \quad (2)$$

where  $\mathbf{a}(\cdot, \cdot)$  and  $\mathbf{b}(\cdot, \cdot)$  are bi-linear forms, while  $c_1(\cdot)$  and  $c_2(\cdot)$  are linear forms. According to the problem at hand, the unknowns  $\mathbf{u}$  and  $p$  represent different physical quantities and, consequently, the function spaces  $\mathbf{V}$  and  $Q$  have to be chosen accordingly.

### 2.1 Mixed-form elliptic equation

Let us consider the continuous spaces

$$\mathbf{V}^D(\Omega) := \left\{ \mathbf{v} \in H(\operatorname{div}; \Omega) : \mathbf{v} \cdot \mathbf{n} = u_N \text{ on } \partial\Omega \right\},$$

$$Q^{\mathcal{D}}(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, d\Omega = 0 \right\},$$

and the linear/bi-linear forms:

$$\begin{aligned} \mathbf{a}^{\mathcal{D}}(\mathbf{v}, \mathbf{w}) : \mathbf{V}^{\mathcal{D}} \times \mathbf{V}^{\mathcal{D}} &\rightarrow \mathbb{R}, & \mathbf{a}^{\mathcal{D}}(\mathbf{v}, \mathbf{w}) &:= \int_{\Omega} \nu(x) \mathbf{v} \cdot \mathbf{w} \, d\Omega, \\ \mathbf{b}^{\mathcal{D}}(\mathbf{v}, q) : \mathbf{V}^{\mathcal{D}} \times Q^{\mathcal{D}} &\rightarrow \mathbb{R}, & \mathbf{b}^{\mathcal{D}}(\mathbf{v}, q) &:= - \int_{\Omega} q \operatorname{div}(\mathbf{v}) \, d\Omega, \\ c_2^{\mathcal{D}}(q) : Q^{\mathcal{D}} &\rightarrow \mathbb{R}, & c_2^{\mathcal{D}}(q) &:= - \int_{\Omega} f q \, d\Omega, \end{aligned} \quad (3)$$

where  $c_1^{\mathcal{D}}(\mathbf{v})$  is identically zero,  $\nu(x)$  is a positive piece-wise constant scalar function and  $f \in L^2(\Omega)$ . Such problem arises in the simulation of multi-phase incompressible flow through porous media. In this framework, we will consider the virtual element spaces introduced in [13].

## 2.2 Stokes problem

For such problem, we consider the following function spaces

$$\mathbf{V}^{\mathcal{S}}(\Omega) := [H_0^1(\Omega)]^3 \quad \text{and} \quad Q^{\mathcal{S}}(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, d\Omega = 0 \right\}.$$

Then, we have the following forms

$$\begin{aligned} \mathbf{a}^{\mathcal{S}}(\mathbf{v}, \mathbf{w}) : \mathbf{V}^{\mathcal{S}} \times \mathbf{V}^{\mathcal{S}} &\rightarrow \mathbb{R}, & \mathbf{a}^{\mathcal{S}}(\mathbf{v}, \mathbf{w}) &:= \int_{\Omega} \nu(x) \varepsilon(\mathbf{v}) : \varepsilon(\mathbf{w}) \, d\Omega, \\ \mathbf{b}^{\mathcal{S}}(\mathbf{v}, q) : \mathbf{V}^{\mathcal{S}} \times Q^{\mathcal{S}} &\rightarrow \mathbb{R}, & \mathbf{b}^{\mathcal{S}}(\mathbf{v}, q) &:= \int_{\Omega} q \operatorname{div}(\mathbf{v}) \, d\Omega, \\ c_1^{\mathcal{S}}(\mathbf{v}) : \mathbf{V}^{\mathcal{S}} &\rightarrow \mathbb{R}, & c_1^{\mathcal{S}}(\mathbf{v}) &:= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega, \end{aligned}$$

where  $c_2^{\mathcal{S}}(\mathbf{v})$  is identically zero,  $\mathbf{f} \in [L^2(\Omega)]^3$ . In this paper, we will use the virtual element discretization proposed in [7] and we refer the reader to [7] and the references therein for a more detailed description of the method.

### 3 Virtual element approximation

Let  $\Omega_h$  be a polyhedral discretization of a domain  $\Omega \subset \mathbb{R}^3$ . To solve one of the problems defined in Section 2, we follow a standard VEM approach, i.e., we define local spaces on each polyhedron  $P \in \Omega_h$  and then we glue them together to obtain the global space. In the following, we describe this procedure in the context of the mixed-form elliptic equation. For the Stokes problem, we refer the reader to [7].

#### 3.1 Virtual element spaces for a mixed-form elliptic problem

To discretize the space  $\mathbf{V}^{\mathcal{D}}(\Omega)$  we consider the virtual element space introduced in [13]. As we said before, we focus on the definition of such space in a polyhedron  $P$  and, for simplicity, we consider only degree  $k = 1$ . Given a polyhedron  $P$ , the velocity field is defined as

$$\mathbf{V}_h^{\mathcal{D}}(P) := \left\{ \mathbf{v}_h \in H(\operatorname{div}; P) \cap H(\mathbf{curl}; P) : \mathbf{v}_h \cdot \mathbf{n}_F \in \mathbb{P}_1(F) \quad \forall F \in \partial P, \right. \\ \left. \operatorname{div}(\mathbf{v}_h) \in \mathbb{P}_0(P), \quad \mathbf{curl}(\mathbf{v}_h) \in [\mathbb{P}_0(P)]^3 \right\}. \quad (4)$$

To uniquely define a function inside such a space, we introduce the following degrees of freedom:

- normal face moments

$$\int_F (\mathbf{v}_h \cdot \mathbf{n}) p_1 \, dF \quad \forall p_1 \in \mathbb{P}_1(F) \quad \text{and} \quad \forall F \in \partial P.$$

- internal cross moments

$$\int_P \mathbf{v}_h \cdot (\mathbf{x} \wedge \mathbf{p}_0) \, dP \quad \forall \mathbf{p}_0 \in [\mathbb{P}_0(P)]^3,$$

where  $\mathbf{x} = (x, y, z)^t$ .

On the other hand, to get a numerical approximation of  $Q^{\mathcal{D}}(\Omega)$ , we use the space of element-wise constant polynomials, i.e.,

$$Q_h^{\mathcal{D}}(P) := \left\{ q_h \in L^2(P) : q_h \in \mathbb{P}_0(P) \right\}.$$

In such a space, a function is determined by one degree of freedom. In the virtual element framework, we take the following degree of freedom:

- the internal moment

$$\int_P q_h \, dP. \quad (5)$$

Although a function  $\mathbf{v}_h$  is virtual, it is possible to compute its divergence and the polynomial defined on each face  $F$  [13]. Furthermore, in order to assemble the linear system, we have to define the  $L^2$ -projection operator  $\mathbf{\Pi}^0 : \mathbf{V}_h^{\mathcal{D}}(P) \rightarrow [\mathbb{P}_1(P)]^3$  such that

$$\int_P \mathbf{\Pi}^0 \mathbf{v}_h \cdot \mathbf{p}_1 \, dP = \int_P \mathbf{v}_h \cdot \mathbf{p}_1 \, dP. \quad (6)$$

To get such a projection operator, we have to be able to compute all the quantities in Equation (6). The left-hand sides are integrals of polynomials inside  $P$  so they are computable. The right-hand side involves the virtual function  $\mathbf{v}_h$  and a-priori is not computable. However, starting from the degrees of freedom and the polynomial identity

$$\mathbf{p}_1 = \nabla p_2 + \mathbf{x} \wedge \mathbf{p}_0,$$

we have

$$\begin{aligned} \int_P \mathbf{v}_h \cdot \mathbf{p}_1 \, dP &= \int_P \mathbf{v}_h \cdot (\nabla p_2) \, dP + \int_P \mathbf{v}_h \cdot (\mathbf{x} \wedge \mathbf{p}_0) \, dP \\ &= - \int_P \operatorname{div}(\mathbf{v}_h) p_2 \, dP + \int_{\partial P} (\mathbf{v}_h \cdot \mathbf{n}) p_2 \, dF + \int_P \mathbf{v}_h \cdot (\mathbf{x} \wedge \mathbf{p}_0) \, dP \\ &= - \int_P \operatorname{div}(\mathbf{v}_h) p_2 \, dP + \sum_{F \in \partial P} \int_F (\mathbf{v}_h \cdot \mathbf{n}_F) p_2 \, dF + \int_P \mathbf{v}_h \cdot (\mathbf{x} \wedge \mathbf{p}_0) \, dP. \end{aligned}$$

Since we can get both the polynomials  $(\mathbf{v}_h \cdot \mathbf{n}_F)$  on each face and the constant divergence of  $\mathbf{v}_h$  via the degrees of freedom, we can compute the right-hand side and the projection operator  $\mathbf{\Pi}^0$  as well.

### 3.2 Operators for a mixed-form elliptic problem

To proceed with the discretization of a mixed-form elliptic problem, we introduce some linear/bi-linear operators that are the discrete counterparts of the ones defined in Equation (3). For this particular problem we define the following local forms:

- flux operator:

$$\mathbf{a}_{h,P}^{\mathcal{D}}(\mathbf{v}_h, \mathbf{w}_h) := \int_P \nu(x) \mathbf{\Pi}^0 \mathbf{v}_h \cdot \mathbf{\Pi}^0 \mathbf{w}_h \, dP + s_P(\mathbf{v}_h - \mathbf{\Pi}^0 \mathbf{v}_h, \mathbf{w}_h - \mathbf{\Pi}^0 \mathbf{w}_h), \quad (7)$$

where  $s_P$  is any symmetric and positive definite bi-linear form which scales as the  $\mathbf{a}(\cdot, \cdot)$ . In this paper we choose the Euclidean scalar product associated with the degrees of freedom of  $\mathbf{V}_h^{\mathcal{D}}(P)$  multiplied by the volume of  $P$  and the value of  $\nu(x)$  at its barycenter, i.e.,

$$s_P(\mathbf{v}_h, \mathbf{w}_h) = \nu(\mathbf{x}_P) |P| \sum_{i=1}^{\#dof_P} dof_i(\mathbf{v}_h) dof_i(\mathbf{w}_h),$$

where  $\#dof_P$  are the number of degrees of freedom associated with a function in  $\mathbf{V}_h^D(P)$  and  $dof_i : \mathbf{V}_h^D(P) \rightarrow \mathbb{R}$  is a linear functional that associates to a function in  $\mathbf{V}_h^D(P)$  the value of its  $i$ -th degree of freedom.

- divergence operator:

$$\mathbf{b}_P^D(\mathbf{v}_h, q_h) := \int_P \operatorname{div}(\mathbf{v}_h) q_h \, dP, \quad (8)$$

- right-hand side operator:

$$c_{2,h,P}^D(q_h) := - \int_P f q_h \, dP.$$

Then, the global one is obtained by the sum of all these local contributions, i.e.,

$$\begin{aligned} \mathbf{a}^D(\mathbf{v}, \mathbf{w}) &\approx \mathbf{a}_h^D(\mathbf{v}_h, \mathbf{w}_h) = \sum_{P \in \Omega_h} \mathbf{a}_{h,P}^D(\mathbf{v}_h, \mathbf{w}_h), \\ \mathbf{b}^D(\mathbf{v}, q) &\approx \mathbf{b}^D(\mathbf{v}_h, q_h) = \sum_{P \in \Omega_h} \mathbf{b}_P^D(\mathbf{v}_h, q_h), \\ c_2^D(q) &\approx c_{2,h}^D(q_h) = \sum_{P \in \Omega_h} c_{2,h,P}^D(q_h). \end{aligned}$$

We can then define the discrete VEM problem as

$$\left\{ \begin{array}{l} \text{find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h^D(\Omega_h) \times Q_h^D(\Omega_h) : \\ \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) - \mathbf{b}(\mathbf{v}_h, p_h) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^D(\Omega_h) \\ \mathbf{b}(\mathbf{u}_h, q_h) = \int_{\Omega_h} f q_h \, d\Omega_h \quad \forall q_h \in Q_h^D(\Omega_h), \end{array} \right. \quad (9)$$

or equivalently in matrix form

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}_h \end{bmatrix}. \quad (10)$$

## 4 Balancing Domain Decomposition by Constraints solver

In this work, we will consider the extension of Balancing Domain Decomposition by Constraints (BDDC) methods to saddle point problems with discontinuous pressure spaces, first presented in [17, 18], and later extended to unstructured finite element computations and arbitrary number of levels in [20]. In what follows, we

will present a concise description of their theoretical aspects by focusing on the VEM discretization; for further algorithmic details, see [20].

#### 4.1 Non-overlapping domain decomposition

We decompose  $\Omega_h$  into  $N$  non-overlapping connected subdomains

$$\overline{\Omega}_h = \bigcup_{i=1}^N \overline{\Omega}_i, \quad \Gamma = \bigcup_{i \neq j} \partial\Omega_j \cap \partial\Omega_i,$$

where each  $\Omega_i$  is a collection of polyhedrons, and where  $\Gamma$  denotes the interface among subdomains.

We then consider the splitting of  $\mathbf{V}_h(\Omega_h)$  into a direct sum of two subspaces  $\mathbf{V}_I \oplus \mathbf{V}_\Gamma$ .  $\mathbf{V}_I$  contains virtual functions identified by degrees of freedom that are associated to only one of the  $\Omega_i$ , whereas the virtual functions in  $\mathbf{V}_\Gamma$  are those associated with degrees of freedom shared by two subdomains, see Section 3. We will use the term subdomain face to identify the union of the polyhedral faces of the mesh that are shared by two given subdomains.

Considering a change of basis of the pressure approximation space

$$Q_h = Q_I \oplus Q_0, \quad Q_0 = \prod_{i=1}^N \{q_h \in Q_h(\Omega_i) \mid q_h \text{ constant in } \Omega_i\},$$

and the computability of  $\mathbf{b}(\mathbf{u}_h, q_h)$ , we can use the divergence theorem and rewrite Equation (10) as

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Gamma} & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{II}^T & B_{II}^T & A_{I\Gamma} & B_{0\Gamma}^T \\ 0 & 0 & B_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{p}_I \\ \mathbf{u}_\Gamma \\ \mathbf{p}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}_\Gamma \\ 0 \\ \mathbf{f}_0 \end{bmatrix}. \quad (11)$$

Our preconditioning strategy considers a block factorization of the above linear system

$$\begin{bmatrix} I & -K_{II}^{-1}K_{I\Gamma} \\ 0 & I \end{bmatrix} \begin{bmatrix} K_{II}^{-1} & 0 \\ 0 & S^\dagger \end{bmatrix} \begin{bmatrix} I & 0 \\ -K_{I\Gamma}^T K_{II}^{-1} & I \end{bmatrix}, \quad (12)$$

where

$$K_{I\Gamma} = \begin{bmatrix} A_{I\Gamma} & 0 \\ B_{I\Gamma} & 0 \end{bmatrix}, \quad K_{II} = \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix},$$

$$S_\Gamma = A_{I\Gamma} - \begin{bmatrix} A_{I\Gamma} \\ B_{I\Gamma} \end{bmatrix}^T \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix} \begin{bmatrix} A_{I\Gamma} \\ B_{I\Gamma} \end{bmatrix}, \quad S = \begin{bmatrix} S_\Gamma & B_{0\Gamma}^T \\ B_{0\Gamma} & 0 \end{bmatrix}. \quad (13)$$

By Sylvester's law of inertia, it can be shown that the velocity Schur complement  $S_\Gamma$  is positive definite. The operator  $K_{II}^{-1}$  is well defined since the space  $Q_I$  does not contain any subdomain constant function. On the other hand, the saddle point Schur complement  $S$  is singular, with a kernel spanned by the representation of the pressure constant function in  $Q_0$ . For the definition of its pseudo-inverse, we require that the application of  $S^\dagger$  will have a null component in such a kernel.

Within the preconditioner, the pseudo-inverse of the saddle point Schur complement  $S^\dagger$  is replaced by the action of a suitable preconditioner, resulting in the preconditioner

$$M^{-1} = \begin{bmatrix} I - K_{II}^{-1}K_{I\Gamma} \\ 0 & I \end{bmatrix} \begin{bmatrix} K_{II}^{-1} & 0 \\ 0 & M_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -K_{I\Gamma}^T K_{II}^{-1} & I \end{bmatrix}. \quad (14)$$

Since subdomain interior problems are solved exactly, the spectrum of the global operator preconditioned by  $M^{-1}$  and the spectrum of  $M_\Gamma^{-1}S$  are the same.

## 4.2 BDDC methods for saddle point Schur complement

BDDC methods construct preconditioners for the interface Schur complement operator  $S$  by relaxing continuity requirements of the operator; specifically, these methods are completely characterized by:

- the definition of a partially assembled space  $\tilde{\mathbf{V}}_\Gamma$ , direct sum of a primal, continuous, space  $\mathbf{V}_\Pi$  and a dual, discontinuous space  $\mathbf{V}_\Delta$ , such that  $\mathbf{V}_\Gamma \subset \tilde{\mathbf{V}}_\Gamma = \mathbf{V}_\Delta \oplus \mathbf{V}_\Pi$
- a scaling operator  $D : \mathbf{V}_\Gamma \rightarrow \tilde{\mathbf{V}}_\Gamma$  such that  $D'D = I$ ; here  $D'$  denotes the adjoint of  $D$ , and it will be referred to as averaging operator.

In this work we use the so-called deluxe variant of the scaling operator; see [14, 10] and the references therein.

A crucial observation for the construction of our preconditioner is that the saddle point problem  $S$  is symmetric positive definite on  $\{\mathbf{V}_\Gamma \cap \ker(B_{0\Gamma})\} \times Q_0$ . Following [18], we then require that the virtual functions in the dual space satisfy the so-called no-net-flux condition

$$\int_{\partial\Omega_i} \mathbf{v}_\Delta \cdot \mathbf{n} = 0, \quad \forall \mathbf{v}_\Delta \in \mathbf{V}_\Delta. \quad (15)$$

Since we have assumed that each  $\Omega_i$  is connected, in our algorithm we will consider one constraint for each subdomain face  $F$

$$\int_F \mathbf{v}_\Delta \cdot \mathbf{n} = 0.$$

The quadrature weights associated with the above constraint can be computed by considering the linear functional  $\mathbf{b}(\cdot, q_0)$  restricted to those virtual functions in  $\mathbf{V}_h$

represented by nonzero degrees of freedom on  $F$  only, and where  $q_0 \in Q_0$  is defined as  $q_0 = 1$  in  $\Omega_i$  and zero otherwise. In turn, the virtual functions in the primal space will be characterized by the continuity of

$$\int_F \mathbf{v}_\Pi \cdot \mathbf{n}, \quad \forall \mathbf{v}_\Pi \in \mathbf{V}_\Pi, \forall F \subset \Gamma. \quad (16)$$

for each subdomain face  $F$ .

Using the divergence theorem, it is easy to see that the VEM problem on  $\{\mathbf{V}_I \oplus \tilde{\mathbf{V}}_\Gamma\} \times \{Q_I \oplus Q_0\}$  can be recast as

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 \\ A_{I\Delta}^T & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} & 0 \\ A_{I\Pi}^T & B_{I\Pi}^T & A_{\Delta\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T \\ 0 & 0 & 0 & B_{0\Pi} & 0 \end{bmatrix}. \quad (17)$$

Eliminating the degrees of freedom of  $\mathbf{V}_I \times Q_I$  we can define a partially assembled Schur complement on  $\tilde{\mathbf{V}}_\Gamma \times Q_0$

$$\begin{bmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} & 0 \\ S_{\Delta\Pi}^T & S_{\Pi\Pi} & B_{0\Pi}^T \\ 0 & B_{0\Pi} & 0 \end{bmatrix}, \quad \tilde{S}_\Gamma = \begin{bmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Delta\Pi}^T & S_{\Pi\Pi} \end{bmatrix}$$

and define the BDDC preconditioner as

$$M_\Gamma^{-1} = D \tilde{S}_\Gamma^\dagger D. \quad (18)$$

By using block factorization arguments, the action of  $\tilde{S}_\Gamma^\dagger$  can be implemented as additive combination of uncoupled subdomain solvers defined on  $\{\mathbf{V}_I \oplus \mathbf{V}_\Delta\} \times Q_I$  and the solution of a coarse saddle point problem on  $\mathbf{V}_\Pi \times Q_0$  which inherits the kernel from  $S$ .

It is easy to see that the preconditioned operator  $M_\Gamma^{-1}S$  is positive definite on  $\{\mathbf{V}_\Gamma \cap \ker(B_{0\Gamma})\} \times Q_0$  by requiring that the average operator preserves fluxes (16) of the virtual functions in  $\tilde{\mathbf{V}}_\Gamma$  [20, Lemma 3.5]. We can then solve the VEM linear system (10) by splitting the velocity solution as  $\mathbf{u}_h = \mathbf{u}_h^* + \bar{\mathbf{u}}_h$ , where  $\mathbf{u}_h^*$  is defined by its components in  $\mathbf{V}_\Gamma$  and  $\mathbf{V}_I$  as

$$\begin{bmatrix} \mathbf{u}_{h,\Gamma}^* \\ * \end{bmatrix} = M_\Gamma^{-1} \begin{bmatrix} 0 \\ \mathbf{f}_{h,0} \end{bmatrix}, \quad (\mathbf{u}_h^*)_I = -K_{II}^{-1} K_{I\Gamma} \mathbf{u}_{h,\Gamma}^*,$$

and use conjugate gradient iterations to solve the deflated system

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} -A\mathbf{u}_h^* \\ \mathbf{f}_h - B\mathbf{u}_h^* \end{bmatrix}. \quad (19)$$

Convergence rate estimates for BDDC preconditioned conjugate gradient methods are completely characterized by the norm of the average operator  $\|D'\|_{\tilde{\mathcal{S}}_F}$ . Unravelling the expression for  $\|D'\|_{\tilde{\mathcal{S}}_F}$ , we can devise generalized eigenvalue problems, defined on each subdomain face  $F$ , to enrich the minimal primal space  $\mathbf{V}_\Pi$  characterized by Eq. (16). In particular, given any  $\nu_{tol} \in [1, \infty)$ , we can algebraically construct an enriched primal space such that condition number of the preconditioned system will be bounded from above by  $\nu_{tol}$  times a constant independent on  $h$ ,  $H$  and  $N$ .

## 5 Numerical results

In this Section, we report numerical results to validate the adaptive BDDC algorithm for solving our two model problems, i.e., the mixed-form elliptic equation and the Stokes equations using VEM discretizations. We will consider Voronoi (CVT) meshes, consisting of elements of general polyhedral shape. In what follows, we will denote by  $k$  the degree of approximation of the VEM discretization.

Our parallel solver is based on the Portable and Extensible Toolkit for Scientific computing (PETSc) [4], which is built on top of the MPI standard. In our code, each subdomain is assigned to a different MPI process. For details on the implementation of the BDDC method in PETSc, see [19, 20].

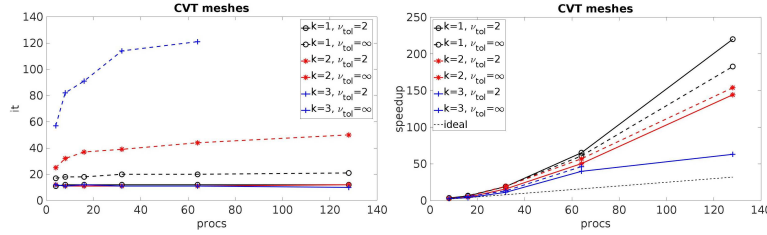
The BDDC algorithms are employed as preconditioners for the conjugate gradient (CG) method, using as a stopping criterion a  $10^{-8}$  reduction of the  $l^2$  norm of the relative residual. We will use  $\nu_{tol} = \infty$  to denote the BDDC algorithm with minimal primal space consisting only of functions with continuous normal fluxes across the subdomain interface.

All the numerical tests presented in the following have been performed on the Linux cluster INDACO ([www.indaco.unimi.it](http://www.indaco.unimi.it)) of the University of Milan, constituted by 16 nodes, each carrying 2 INTEL XEON E5-2683 V4 processors at 2.1 GHz, with 16 cores each.

### 5.1 Strong scalability for the mixed-form elliptic equation

We increase the number of processors from 4 to 128, while keeping fixed the global number of degrees of freedom (dofs). The CVT mesh adopted consists of 16000 elements and 388042 dofs for  $k = 1$ , 8000 elements and 465720 dofs for  $k = 2$  and 4000 elements and 465720 dofs for  $k = 3$ .

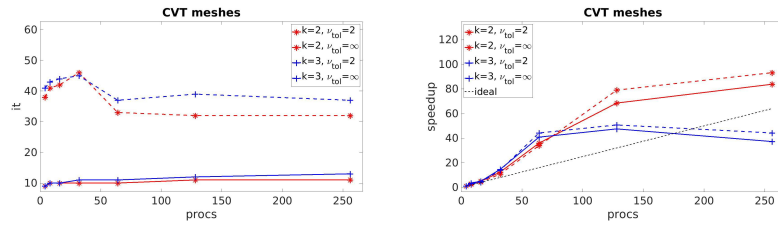
The results reported in Figure 1 show the scalability of the adaptive BDDC solver, both in terms of CG iterations and CPU times, irrespective of the degree of approximation. The minimal primal space solver exhibits a good scalability for  $k = 1$ , however for  $k = 2$  and  $k = 3$  the CG iterations slightly increase with the number of processors.



**Fig. 1** Mixed-form elliptic equation, strong scalability. Plots of conjugate gradient iterations (left) and parallel speedup (right) with respect to the number of processors on **CVT** meshes for adaptive ( $\nu_{tol} = 2$ , solid lines) and minimal primal space ( $\nu_{tol} = \infty$ , dashed lines) BDDC preconditioners, with different degrees  $k$  of VEM approximation.

## 5.2 Strong scalability for the Stokes problem

We keep fixed the global number of dofs and the degree of the VEM approximation  $k$ , while we increase the number of processors from 4 to 256. The **CVT** mesh adopted consists of 4000 elements and 311155 dofs for  $k = 2$  and 1000 elements and 163093 dofs for  $k = 3$ .



**Fig. 2** Stokes problem, strong scalability. Plots of conjugate gradient iterations (left) and parallel speedup (right) with respect to the number of processors on **CVT** meshes for adaptive ( $\nu_{tol} = 2$ , solid lines) and minimal primal space ( $\nu_{tol} = \infty$ , dashed lines) BDDC preconditioners, with different degrees  $k$  of VEM approximation.

In Figure 2, we plot the number of iterations and the parallel speedup. The adaptive BDDC method ( $\nu_{tol} = 2$ ) results algorithmically scalable since the number of CG iterations remains bounded and the solution time decreases as the number of the processors increases. The parallel speedup shows a superlinear rate and improves only up to the point where communication times start to dominate, as usual in the strong scaling tests of domain decomposition methods where local problems are solved using direct factorizations. Also, the minimal coarse space results are scalable, with the same behavior as the adaptive BDDC.

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