

Schwarz Modulus Based Matrix Splittings with Minimal Polynomial Extrapolation Acceleration for Linear Complementarity Problems Arising from American Option Pricing

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1 Introduction

Options are one of the most fundamental financial derivative instruments in modern markets. They give the holder the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a fixed strike price, either on or before a specified expiration date. The evaluation of options has long been an active research field. One of the most famous models for option pricing is the Black–Scholes model, introduced in 1973 by Black and Scholes [2]. Based on when the options can be exercised, they are classified as European and American options. European options can only be exercised at expiration, and their prices can be obtained analytically by solving the Black–Scholes equation, which means solving a diffusive boundary value problem. In contrast, American options may be exercised at any time before expiration. This early exercise feature imposes an additional constraint on the option value, transforming the Black–Scholes model into a free boundary value problem. Due to this complexity, American options generally do not have closed-form solutions, and numerical methods must be used.

A simple numerical method for pricing American options is the binomial method proposed by Cox, Ross and Rubinstein [6]. It converges however slowly and is difficult to extend to higher dimensions. In contrast, the finite difference method (FDM) [3, 4] is widely used, offering both high accuracy and extensibility to higher dimensions. Discretizing the Black–Scholes model in both space and time, the FDM transforms the continuous partial differential equation (PDE) into a discrete algebraic system. For American options, the resulting discretized model can be reformulated as a linear

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complementarity problem (LCP) [10], whose efficient numerical solution is an active research topic. Recently, the modulus-based matrix splitting (MMS) method [1] has been introduced as an efficient approach for solving LCPs, due to good numerical properties and convergence. One first uses a transformation to reformulate the LCP into an equivalent implicit fixed-point equation, and then applies a modulus-based matrix splitting strategy to construct an iterative scheme. Compared with conventional methods, such as the projected successive over relaxation method [7] and the fixed-point method in [14], modulus based methods avoid projection operations at each iteration, reducing computational cost.

To improve the convergence of MMS for American options pricing, we introduce in this paper a new Schwarz modulus based matrix splitting method, and the minimal polynomial extrapolation (MPE), a non-linear vector sequence convergence acceleration technique, which is very much related to Krylov methods in the linear case [12]. In Section 2, we reformulate the Black–Scholes model as a standard LCP using variable transforms and a finite difference discretization. We present in Section 3 the new Schwarz modulus based splitting method and its MPE acceleration, followed in Section 4 by numerical results, and some concluding remarks.

2 Model, transformation and discretization of American options

We denote by $V(S, t)$ the price of an American option and by $G(S)$ the given payoff function. Here, $S > 0$ is the underlying asset value and $t \in [0, T]$ is the time variable with T the expiration time. The Black–Scholes operator \mathcal{L} is defined as $\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta)S \frac{\partial}{\partial S} - r$, where σ is the fixed volatility, r is the constant risk free interest rate, and δ is the continuous dividend yield. Due to the early exercise possibility of American options, $V(S, t)$ should be larger than $G(S)$ in order to avoid arbitrage possibilities. Based on a standard no arbitrage argument, $V(S, t)$ satisfies the complementarity problem

$$\mathcal{L}V \leq 0, \quad V(S, t) \geq G(S), \quad \text{and} \quad \mathcal{L}V \cdot (V - G) = 0, \quad (1)$$

with the final condition $V(S, T) = G(S)$. For a fixed strike price K , the payoff function $G(S)$ is defined as

$$\text{call option: } G(S) = \max\{S - K, 0\}, \quad \text{put option: } G(S) = \max\{K - S, 0\}. \quad (2)$$

To complete the problem, we have the boundary conditions $\lim_{S \rightarrow 0} V(S, t) = K$ and $\lim_{S \rightarrow \infty} V(S, t) = 0$ for put options, and $\lim_{S \rightarrow 0} V(S, t) = 0$ and $\lim_{S \rightarrow \infty} V(S, t) \sim S - K \exp(-r(T - t))$ for call options. Note that the asset value S in practice is in a range of (S_{\min}, S_{\max}) , where typically Dirichlet boundary conditions are imposed.

A change of variables can transform the Black–Scholes equation into a standard heat equation. This transformation simplifies the structure of the Black–Scholes equation, and improves the stability properties for numerical methods. As shown in [15],

this transformation leads to a flattened eigenvalue distribution of the discretized PDE operator. Hence, numerical schemes applied to the transformed heat equation achieve better stability behavior compared to those applied directly to the original Black–Scholes equation. For this reason, we introduce the change of variables

$$t = T - \frac{2\tau}{\sigma^2}, \quad S = Ke^x, \quad \text{and} \quad V(S, t) = Ke^{\alpha x + \beta \tau} u(x, \tau), \quad (3)$$

where $\alpha = -(h_\delta - 1)/2$, $\beta = -(h_\delta - 1)^2/4 - h$, $h = 2r/\sigma^2$, and $h_\delta = 2(r - \delta)/\sigma^2$. The original time interval $t \in [0, T]$ now becomes $\tau \in [0, \sigma^2 T/2]$, where the new variable τ propagates backward with respect to t , and represents the remaining lifetime of the American option. For the asset value S , the logarithmic transformation maps the original semi-infinite domain $S \in (0, \infty)$ to the entire real line $x \in (-\infty, \infty)$. Using the change of variables (3), we can then derive a complementarity problem for the new unknown $u(x, \tau)$,

$$-\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2} \leq 0, \quad u(x, \tau) \geq g(x, \tau), \quad \text{and} \quad \left(-\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2}\right)(u - g) = 0, \quad (4)$$

which is equivalent to the Black–Scholes complementarity problem (1). Furthermore, the payoff function (2) now becomes

$$g(x, \tau) = \begin{cases} e^{\frac{\tau}{4}((h_\delta-1)^2+4h)} \max \{e^{\frac{x}{2}(h_\delta+1)} - e^{\frac{x}{2}(h_\delta-1)}, 0\}, & \text{for a call option,} \\ e^{\frac{\tau}{4}((h_\delta-1)^2+4h)} \max \{e^{\frac{x}{2}(h_\delta-1)} - e^{\frac{x}{2}(h_\delta+1)}, 0\}, & \text{for a put option.} \end{cases} \quad (5)$$

The final condition $V(S, T) = G(S)$ becomes an initial condition $u(x, 0) = g(x, 0)$, and the boundary conditions of $V(S, t)$ are now $\lim_{x \rightarrow \pm\infty} u(x, \tau) = \lim_{x \rightarrow \pm\infty} g(x, \tau)$. Once we have solved for $u(x, \tau)$, we can recover the original solution $V(S, t)$ using the change of variables (3).

To apply the FDM to discretize the transformed complementarity problem (4), we replace the infinite spatial domain $(-\infty, \infty)$ by a truncated interval (a, b) with $a < 0 < b$. Such a truncated interval is chosen based on the economical relevance of the asset value S , e.g., prices beyond which the option value is negligible. We denote this range of the asset value by (S_{\min}, S_{\max}) , and choose a, b such that $Ke^a \leq S_{\min}$, $Ke^b \geq S_{\max}$ to guarantee numerical accuracy within (S_{\min}, S_{\max}) . Note that the choice of the interval has a direct impact on the accuracy of the numerical approximation. In particular, a should be chosen sufficiently small and b sufficiently large to keep the truncation error within an acceptable tolerance. However, taking a too small or b too large increases computational cost. Thus, there is a trade-off between the numerical accuracy and the computational cost. One could also use absorbing boundary conditions for better truncation properties, but this is not well explored; practical choices of the truncation interval can be found, e.g., in [11].

We apply a uniform grid to discretize the space-time domain $(a, b) \times (0, \sigma^2 T/2]$. Let $\Delta x = \frac{b-a}{n}$ and $\Delta \tau = \sigma^2 T/(2m)$ denote the mesh sizes in x and τ direction. Define the grid points as $x_i = a + i\Delta x$, $i = 0, \dots, n$, and $\tau_j = j\Delta \tau$, $j = 0, \dots, m$, and denote

the approximate solution by $u_i^j \approx u(x_i, \tau_j)$. A θ -weighted FDM applied to the first inequality in (4) gives $u_i^{j+1} - \lambda\theta(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}) \geq u_i^j + \lambda(1-\theta)(u_{i+1}^j - 2u_i^j + u_{i-1}^j)$, for $i = 1, \dots, n-1$, $j = 0, \dots, m-1$, $\lambda := \Delta\tau/\Delta x^2$, and $\theta \in (0, 1)$. In the matrix-vector form, we have $A\mathbf{u}^{j+1} \geq B\mathbf{u}^j + \mathbf{b}^{j+1} =: \mathbf{r}^j$, where $A = I_{n-1} + \lambda\theta T$, $B = I_{n-1} - \lambda(1-\theta)T$ and $\mathbf{u}^j = [u_1^j, \dots, u_{n-1}^j]^\top$. Here, $I_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$ denotes the identity matrix, $T \in \mathbb{R}^{(n-1) \times (n-1)}$ is a tridiagonal matrix and $\mathbf{b}^{j+1} \in \mathbb{R}^{n-1}$ is a vector, namely

$$T = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \mathbf{b}^{j+1} = \begin{bmatrix} \lambda(1-\theta)g_0^j + \lambda\theta g_0^{j+1} \\ 0 \\ \vdots \\ 0 \\ \lambda(1-\theta)g_n^j + \lambda\theta g_n^{j+1} \end{bmatrix},$$

with $g_i^j \approx g(x_i, \tau_j)$. Then, the FDM discretized complementarity problem (4) becomes a sequence of complementarity problems: $A\mathbf{u}^{j+1} - \mathbf{r}^j \geq 0$, $\mathbf{u}^{j+1} \geq \mathbf{g}^{j+1}$, and $(A\mathbf{u}^{j+1} - \mathbf{r}^j)^\top (\mathbf{u}^{j+1} - \mathbf{g}^{j+1}) = 0$, with the initial condition $u_i^0 = g_i^0$, $i = 1, \dots, n-1$, and boundary conditions $u_0^j = g_0^j$, $u_n^j = g_n^j$, $j = 0, \dots, m-1$. For computational convenience, we can write it in the standard form of an LCP. To achieve this, we define $\mathbf{z} := \mathbf{u}^{j+1} - \mathbf{g}^{j+1}$ and $\mathbf{q} := A\mathbf{g}^{j+1} - \mathbf{r}^j$ and re-write the early exercise constraint $\mathbf{u}^{j+1} \geq \mathbf{g}^{j+1}$ as a nonnegativity condition $\mathbf{u}^{j+1} - \mathbf{g}^{j+1} \geq 0$. This gives the standard LCP form

$$A\mathbf{z} + \mathbf{q} \geq 0, \quad \mathbf{z} \geq 0 \quad \text{and} \quad (A\mathbf{z} + \mathbf{q})^\top \mathbf{z} = 0. \quad (6)$$

Note that the vector \mathbf{q} is given, since \mathbf{g}^{j+1} is given and \mathbf{r}^j is known at the j th time step. The only unknown now is \mathbf{z} , which is the solution of the LCP (6). In other words, one needs to solve an LCP at each time step, which motivates the development of efficient numerical methods.

3 Schwarz modulus based matrix splitting and minimal polynomial extrapolation

We now present numerical methods for solving the LCP (6). The classical MMS consists of applying a modulus transformation to preserve both the complementarity and nonnegativity conditions in (6). It has been shown in [1, Theorem 2.1] that the LCP (6) is equivalent to the fixed point problem: find $\mathbf{y} \in \mathbb{R}^{n-1}$ such that

$$(M + \Omega)\mathbf{y} = N\mathbf{y} + (\Omega - A)|\mathbf{y}| - \eta\mathbf{q}, \quad (7)$$

where $A = M - N$ is a matrix splitting of A , typically Gauss-Seidel, $\Omega \in \mathbb{R}^{(n-1) \times (n-1)}$ is a positive diagonal matrix, and $\eta > 0$ is a constant. A detailed discussion on the choices of η and Ω is given in [1, Section 3]. In particular, if \mathbf{y} satisfies (7),

$z = (|y| + y)/\eta$ solves the LCP (6). On the other hand, if z is a solution of the LCP (6), then $y = \eta\Omega^{-1}((\Omega - A)z - q)/2$ satisfies the fixed-point problem (7). Hence, one can solve the fixed point problem (7) and then recover the solution for the LCP (6) using their equivalence. A natural stationary iteration to solve (7) is given by $(M + \Omega)y_{k+1} = Ny_k + (\Omega - A)|y_k| - \eta q$, with a given initial vector $y_0 \in \mathbb{R}^{n-1}$, and the iteration index $k = 0, 1, \dots$. Once y_{k+1} is computed, the corresponding approximate solution of the original LCP (6) can be recovered using $z_{k+1} = (|y_{k+1}| + y_{k+1})/\eta$.

We introduce now as a new solver an alternating Schwarz method: we decompose the spatial domain (a, b) into subdomains with minimal overlap of one mesh size Δx , which corresponds to a block Gauss–Seidel method, see e.g. [8]. This decomposition permits an arbitrary number of overlapping subdomains. For our problem, we observe from our numerical experiments that a two-subdomain decomposition gives the best performance. So we focus here on two subdomains, which gives

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & -A_{12} \\ 0 & 0 \end{pmatrix},$$

where A_{11} is the matrix corresponding to the discrete problem in the spatial subdomain $(a, \Delta x)$ to the left of the origin (recall that $a < 0$) with overlap Δx , and A_{22} is the corresponding matrix in the subdomain $(0, b)$ to the right of the origin. The matrix A_{12} corresponds to the Dirichlet transmission condition at the interface $x = \Delta x$, similarly A_{21} is the matrix for the Dirichlet transmission condition at $x = 0$. For comparison, we also consider the classical point-wise Gauss–Seidel splitting with $M = D + L$ and $N = -U$, where D , L and U are the diagonal, strictly lower triangular, and strictly upper triangular matrices.

Note that we need to solve at each time step an LCP problem (6), and when there are many time steps, the total iteration number can be very large to solve all the fixed point problems (7). To further accelerate this process, we consider Modified Polynomial Extrapolation (MPE) introduced by Cabay and Jackson [5], which is a powerful acceleration technique for improving the convergence of non-linear vector sequences. It is very effective for accelerating fixed-point iterations used to solve linear or nonlinear systems of equations, especially those arising in the discrete solution of continuum problems, and is related to Krylov methods in the linear case, see e.g. [9, Section 11.6 and 11.7] and [12].

For a sequence of vectors $y_0, y_1, y_2, \dots, y_{k+1}$ generated by the MMS method, we first define the sequence of the difference between two vectors as $u_i := y_{i+1} - y_i$ for $i = 0, 1, \dots, k$, and put them into the $(n-1) \times (k+1)$ matrices $U_k := (u_0, u_1, \dots, u_k)$. The MPE approximation s_k is given by the linear combination $s_k = \sum_{i=0}^k \gamma_i y_i$, where the coefficients γ_i are determined such that $U_k \gamma \approx 0$ and $\sum_{i=0}^k \gamma_i = 1$, computed in two steps: First, solve the linear system $U_{k-1} c \approx -u_k$ as a least squares problem, where $c = (c_0, c_1, \dots, c_{k-1})^T$. Second, set $c_k = 1$, and compute the scaled coefficients by $\gamma_i = c_i / \sum_{j=0}^k c_j$ for $i = 0, 1, \dots, k$. When $\sum_{j=0}^k c_j = 0$, the extrapolated vector s_k does not exist.

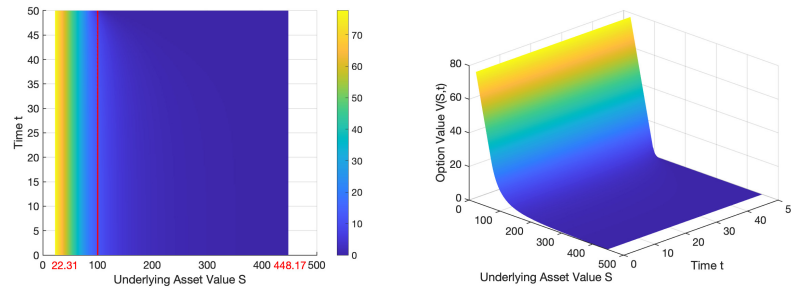


Fig. 1 Option value V as a function of the asset value S and time t using our new Schwarz splitting with a mesh size $(\Delta x, \Delta \tau) = (2^{-7}, 2^{-7})$. Left: 2D view. Right: 3D view.

4 Numerical experiments and discussion

We illustrate now the numerical performance of our new Schwarz MMS and the new MPE acceleration for solving the LCP arising from American option pricing. We consider an American put option on a single underlying asset S . We set $K = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0$, and $T = 50$. For the FDM discretization, we set $(a, b) = (-1.5, 1.5)$ and use the Crank–Nicolson scheme with $\theta = 1/2$. We show in Fig. 1 the American put option price V as a function of the underlying asset value S and time t , computed using our new Schwarz MPE. The asset value ranges from $S_{\min} = Ke^a \approx 22.31$ to $S_{\max} = Ke^b \approx 448.17$. Within this interval (S_{\min}, S_{\max}) , the option value V can be determined at any time t . In Fig. 1 (left), the option value V decreases steadily as the asset value S increases, consistent with classical option pricing theory. Given the strike price $K = 100$, when S is significantly greater than K , the option value V approaches zero. Fig. 1 (right) further shows that as expiration approaches, the put option value curve becomes steeper with increasing S , emphasizing the accelerated decay of the option value when $S > K$.

In the MMS, we set $\eta = 2$ and $\Omega = 1/2 \text{diag}(A)$, and we test the classical pointwise Gauss–Seidel splitting, denoted by GS, two MPE accelerations, MPE-GS and MPECycle-GS, and the new Schwarz or Block Gauss-Seidel splitting, denoted by BGS, with their MPE accelerated versions, MPE-BGS and MPECycle-BGS. In the first time step, the initial vector \mathbf{y}_0 is set to the prescribed initial condition, *i.e.*, all entries are initialized to 0. For subsequent time steps, the initial vector is taken as the numerical solution obtained from the previous time step. At each time step, the stationary iteration is terminated when the residual satisfies $\|A(|\mathbf{y}_{k+1}| + \mathbf{y}_{k+1}) + \Omega(\mathbf{y}_{k+1} - |\mathbf{y}_{k+1}|) + \eta \mathbf{q}\|_{\infty} < 10^{-6}$.

To compare the performance of the classical MMS and new BGS splitting with and without MPE acceleration, we show in Table 1 the iteration numbers and computational time in seconds obtained with different mesh sizes for a convergence tolerance of 10^{-6} . For each method, we show the total CPU time, total number of iterations over all time steps and the average number of iterations, which is computed by dividing the total iterations over all time steps by the number of time steps. For the

Table 1 Comparison of our new Schwarz splitting strategy (BGS) to classical Gauss-Seidel (GS) and their MPE accelerated versions with different mesh sizes

$(\Delta x, \Delta \tau)$	Method	Total Iterations	Average Iterations	CPU Time
$(2^{-4}, 2^{-4})$	GS	1403	87	0.023839
	MPE-GS	429	27	0.021326
	MPECycle-GS	480	30	0.007024
	BGS	865	54	0.016162
	MPE-BGS	297	19	0.012260
	MPECycle-BGS	363	23	0.005096
$(2^{-5}, 2^{-5})$	GS	5491	171	0.072522
	MPE-GS	1753	55	0.264530
	MPECycle-GS	1746	55	0.027914
	BGS	3168	99	0.058267
	MPE-BGS	936	29	0.066916
	MPECycle-BGS	1072	34	0.022632
$(2^{-6}, 2^{-6})$	GS	21123	330	0.619466
	MPE-GS	6542	102	5.260333
	MPECycle-GS	6368	100	0.230351
	BGS	11666	182	0.418613
	MPE-BGS	2887	45	0.443999
	MPECycle-BGS	3328	52	0.144547

pointwise splitting case, we observe that compared with GS, MPE-GS divides the required iteration number for convergence by a factor of 3.21 on average. However, the performance in terms of the required CPU time is much different. As the mesh is refined, the required CPU time of MPE-GS increases drastically, due to the solving of the least square problem $U_{k-1}\mathbf{c} \approx -\mathbf{u}_k$ at each MPE iteration using the backslash of MATLAB¹, and the proportion of CPU time needed rises sharply from 53.5% to 92.2%. Based on this observation, we apply an improved MPE acceleration method, MPECycle-GS, which consists of using the MPE acceleration only after every N_c iterations of GS. This significantly improves the performance of the MPE acceleration, as it reduces both the frequency and size of solving least square problems, i.e., from solving for all previous iterates to only for the previous N_c iterates. In our tests, $N_c = 15$ gives a good improvement compared with GS, i.e., a speedup of 3.13 in terms of iteration numbers and 2.89 in terms of CPU time. This clearly shows that MPE is capable of a significant acceleration of the non-linear point-wise GS process.

We also see that our new Schwarz splitting variant BGS requires fewer iterations compared with GS. This is expected, since the Schwarz-based BGS takes advantage of the subdomain solves, yielding a stronger global correction at each iteration. The performance of both methods is similar in terms of CPU time. However, by applying MPE to accelerate BGS, we see that the required CPU time of MPE-BGS is comparable to that of BGS, since the proportion of CPU time required for solving the least square problem is around 50% on average, unlike a drastic increase in the point-wise case. When applying MPE every N_c iterations, MPECycle-BGS can

¹ There are faster ways to implement the numerical solving of such least square problems by updating, like for Krylov methods, see e.g., [13], but this is beyond the scope of our manuscript.

further improve the performance with a speedup of 2.95 in terms of iteration numbers and 2.88 in terms of CPU time compared with BGS. Overall, our novel Schwarz-based algorithms outperform the classical splitting methods, which motivates a further investigation of this new algorithm applied to LCP problems.

Due to the early exercise constraint, the American option pricing problem is harder to solve numerically than the more classical European option pricing problem, and one must solve a sequence of linear complementarity problems (LCPs). To do so, we introduced a new Schwarz modulus bases splitting method for solving such LCPs, and further accelerated convergence using Modified Polynomial Extrapolation, a non-linear vector sequence acceleration technique related to Krylov methods in the linear case [12]. Our new Schwarz modulus based provides a brand new avenue for numerically solving the American option pricing problem.

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