Right Nonlinear Preconditioning for the Backward-Facing Step Flow

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1 Introduction

Newton-like methods in their many variants often suffer from residual stagnation or convergence failure when applied to highly nonlinear problems. To enhance global convergence, various continuation techniques have been developed to provide better initial iterates. These include parameter continuation [11], mesh sequencing [17], and pseudo-time-stepping [6]. As an alternative, nonlinear preconditioning – often devised as an inner-outer Newton solver – can enhance the robustness of Newton-type methods. This is achieved by reducing their sensitivity to the initial guess and addressing the system's inherent nonlinear stiffness.

Analogously to linear preconditioning, nonlinear preconditioning techniques overcome challenges posed by unbalanced nonlinearities via left- or right-side nonlinear transformations. Classical left nonlinear preconditioners such as the additive Schwarz preconditioned inexact Newton (ASPIN) [3], the restricted additive Schwarz preconditioned inexact Newton (RASPEN) [7, 9] and the multiplicative Schwarz preconditioned inexact Newton (MSPIN) [15], transform the original system into an equivalent form, yielding a preconditioned system that preserves the solution while demonstrating improved nonlinear convergence, which is then solved by an outer Jacobian-free Newton method [11].

Right nonlinear preconditioners work by recasting the solution space through a nonlinear transformation and solving the problem in this new space. A common

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approach to constructing this transformation is nonlinear elimination (NE) [12], which removes variables deemed problematic before initiating a global Newton iteration. Examples of this approach include the domain-based method Newton-Krylov-Schwarz restricted additive Schwarz (NKS-RAS) [4] and the field-based variants nonlinear-elimination inexact Newton with backtracking (INB-NE) and the active-set reduced-space method with nonlinear elimination (RS-NE) [19, 20]. It should be noted that the selection of "bad" components—those hindering global convergence—is generally problem-dependent, and insufficient removal of such components may fail to rescue subsequent global Newton iterations. Other variants of right nonlinear preconditioning, such as nonlinear dual-primal finite element tearing and interconnecting (FETI-DP) domain decomposition approaches, have proved effective and are also attracting increasing attention; see [10].

In this work, we present a novel right-preconditioned inexact Newton algorithm for the backward-facing step flow problem. Numerical experiments validate the effectiveness of our nonlinear transformation strategy in solving this benchmark flow problem.

2 Stratified flow over a backward-facing step

In this study, we consider a simplified two-dimensional stratified flow over a backward-facing step in an infinite domain, following the configuration described in [13]. The computational domain is defined as $\Omega = (0, 30) \times (-0.5, 0.5)$, and the problem involves four unknown fields: the velocity components (u, v) in the x- and y-directions, the vorticity ω, and the temperature T. The governing equations consist of the nondimensional steady-state Navier–Stokes equations in vorticity–velocity form, coupled with the energy equation.

$$\begin{cases} -\Delta u - \frac{\partial \omega}{\partial y} = 0, \\ -\Delta v + \frac{\partial \omega}{\partial x} = 0, \\ -\frac{1}{Re}\Delta\omega + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \frac{Gr}{Re^2}\frac{\partial T}{\partial x} = 0, \\ -\frac{1}{RePr}\Delta T + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0, \end{cases} \tag{1}$$

where Re, Gr, and Pr represent the Reynolds, Grashof, and Prandtl numbers, respectively. In our simulations, we take $Pr = 1$ and set the Richardson number as $Ri = \frac{Gr}{Re^2} = \frac{9}{16}$. The length of the channel (namely 30 channel widths) is intended to give the exiting flow sufficient development that simple extrapolative outflow conditions apply. The following boundary conditions are imposed:

- Along the left boundary Γ_{left},

$$\begin{cases} u = 24y(0.5 - y), \quad v = 0, \quad T = 2y, \; y \in [0, 0.5], \\ u = 0, \quad v = 0, \quad T_x = 0, \qquad\qquad y \in [-0.5, 0), \end{cases} \qquad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the top boundary Γ_{top},

$$u = 0, \quad v = 0, \quad T = 1, \quad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the bottom boundary Γ_{bottom},

$$u = 0, \quad v = 0, \quad T = 0, \quad \omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

- Along the right boundary Γ_{right},

$$u_x = 0, \quad v_x = 0, \quad T_x = 0, \quad \omega_x = 0.$$

To discretize this problem, we first use the standard second-order central finite difference scheme to discretize the Laplace operator and the first-order partial derivative [18]. For the vorticity boundary conditions, a second-order finite difference scheme using only immediate neighboring points is applied [16]. To approximate the first-order derivatives on the left and right boundaries, the second-order forward and backward difference schemes are used, respectively. The discretization leads to a nonlinear system overall because of the quadratic convective terms for vorticity and internal energy.

3 The Right-Preconditioned Inexact Newton Algorithm

We consider a nonlinear system of algebraic equations $F : D \subset R^n \rightarrow R^n$, where we seek a vector $y^* \in R^n$ such that

$$F(y^*) = 0.$$

In contrast to left nonlinear preconditioning, which operates on the residual, right nonlinear preconditioning transforms the coordinates of the solution, consistent with the conventional interpretation of "right" in linear preconditioning, i.e.,

$$F(y) = 0, \quad y = G(x). \tag{2}$$

Here, the nonlinear transformation $G(x)$ is often associated with a nonlinear elimination (NE) [12] procedure, which requires partitioning both the nonlinear system and the variables into two groups: "bad" and "good". The corresponding restriction matrices R_b and R_g are defined accordingly. For a given vector x, we define the nonlinear operator G through the relation $y = G(x)$, where y as the soluton of $\mathcal{F}(\hat{y}) = R_b F(\hat{y}) + R_g(\hat{y} - x) = 0$. The computational efficiency of the nonlinear elimination preconditioned inexact Newton methods—including the INB-NE approach—is highly dependent on the selection of eliminated variables (i.e., the bad components).

Unlike the nonlinear elimination strategy, we propose a novel nonlinear transformation strategy to define $G(x)$ for the backward-facing step flow, thereby avoiding the need to split variables into "bad" and "good" groups. Based on the Oseen-linearized form [2] given in Equation (1),

$$\begin{cases} -\Delta u - \frac{\partial \omega}{\partial y} = 0, \\ -\Delta v + \frac{\partial \omega}{\partial x} = 0, \\ -\frac{1}{Re}\Delta \omega + u^* \frac{\partial \omega}{\partial x} + v^* \frac{\partial \omega}{\partial y} - \frac{Gr}{Re^2}\frac{\partial T}{\partial x} = 0, \\ -\frac{1}{RePr}\Delta T + u^* \frac{\partial T}{\partial x} + v^* \frac{\partial T}{\partial y} = 0, \end{cases} \tag{3}$$

where (u^*, v^*) is a given velocity field approximation. Let $x^{(k)}$ be an intermediate solution at the k-th Newton iteration, and denote by $x_{i,j}^{(k)} = (u_{i,j}^*, v_{i,j}^*, \omega_{i,j}^*, T_{i,j}^*))$ the component of $x^{(k)}$ at the grid point (x_i, y_j). Substituting $(u_{i,j}^*, v_{i,j}^*)$ as the known velocity into the discretized version of Equation (3) yields the system whose solution defines $G(x^{(k)})$.

Algorithm 1 outlines a right-preconditioned inexact Newton method that incorporates the aforementioned nonlinear transformation strategy with backtracking.

Algorithm 1 Right-Preconditioned Inexact Newton Algorithm (Right-PIN)

Specify the initial guess $x^{(0)} = [u_1^{(0)}, u_2^{(0)}]^T$ and $k = 0$.
while $\|F(x^{(k)})\| > \epsilon_{\text{global-nonlinear-rtol}}\|F(x^{(0)})\|$ and $\|F(x^{(k)})\| > \epsilon_{\text{global-nonlinear-atol}}$ **do**

1. Compute the corrected point $\tilde{x}^{(k)} = G(x^{(k)})$.
2. Find the inexact Newton direction $d^{(k)}$ such that

$$J(\tilde{x}^{(k)})d^{(k)} = -F(\tilde{x}^{(k)})$$

3. Compute the new approximate solution

$$x^{(k+1)} = \tilde{x}^{(k)} + \lambda^{(k)}d^{(k)},$$

where $\lambda^{(k)} \in (0, 1]$ is the damping parameter determined by a backtracking line search along $d^{(k)}$ such that

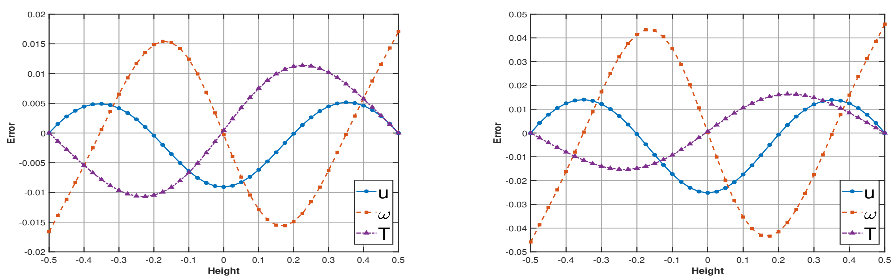
$$\|F(\tilde{x}^{(k)} + \lambda^{(k)}d^{(k)})\| \leqslant \theta\|F(\tilde{x}^{(k)})\|, \quad \theta \in (0, 1).$$

Set $k = k + 1$.
end while

Remark 1 Similar to INB-NE, Algorithm 1 lacks global convergence guarantees. The condition $\|F(x^{(k+1)})\| < \|F(\tilde{x}^{(k)})\|$ is insufficient to guarantee monotonic convergence, as it does not prevent the possibility of $\|F(x^{(k+1)})\| > \|F(x^{(k)})\|$ [14].

4 Numerical Experiments

In this section, we present numerical experiments implemented using the PETSc library [1]. The global nonlinear system is solved using the Inexact Newton Backtracking (INB) method, with absolute and relative tolerances set to 10^{-12} and 10^{-10}, respectively. The resulting Jacobian systems are solved using the generalized minimum residual method (GMRES), preconditioned by the right restricted additive Schwarz (RAS) method [5] with an overlap of 4, where each individual block is solved by the direct LU decomposition. The global linear iterations use absolute and relative tolerances of 10^{-10} and 10^{-6}, respectively. To compute $G(x^{(k)})$, the discretized linear system of Equation (3) is solved inexactly with linear relative tolerance of 10^{-3} offering a trade-off between the accuracy of the preconditioning step and its associated computational cost. A zero initial guess is used for all Newton iterations in numerical tests.



(a) Re = 800, 1201 × 41 mesh (b) Re = 1000, 1201 × 41 mesh

Fig. 1 Vertical profiles of the normalized differences between the computed values and the analytical stratified Poiseuille flow solution at the domain boundary $x = 30$ for the u-velocity component, vorticity and temperature, obtained on a 1201 × 41 mesh at different Reynolds numbers.

Before analyzing the convergence behavior of the solvers, we first verify that the boundary conditions are suitably imposed so as not to significantly perturb the solution within the finite domain. According to Erturk [8], the solution of the two-dimensional steady incompressible Navier-Stokes equations asymptotically approaches a fully developed plane Poiseuille flow sufficiently far downstream of the step. Fig. 1 validates this by comparing the computed profiles of u-velocity, vorticity, and temperature at the outlet boundary ($x = 30$) with the corresponding analytical solution for stratified Poiseuille flow, using a 1201 × 41 mesh at various Reynolds numbers. At $Re = 800$, the computed results agree with the analytical solution within 1% for u-velocity, 1.25% for temperature, and 2% for vorticity. However, at $Re = 1000$, the vorticity error rises to about 5%, suggesting that an outflow boundary at $x = 30$ is inadequate for Reynolds numbers above 1000, owing to the influence of the finite domain. Thus, we restrict the Reynolds number to 1000 in the following tests.

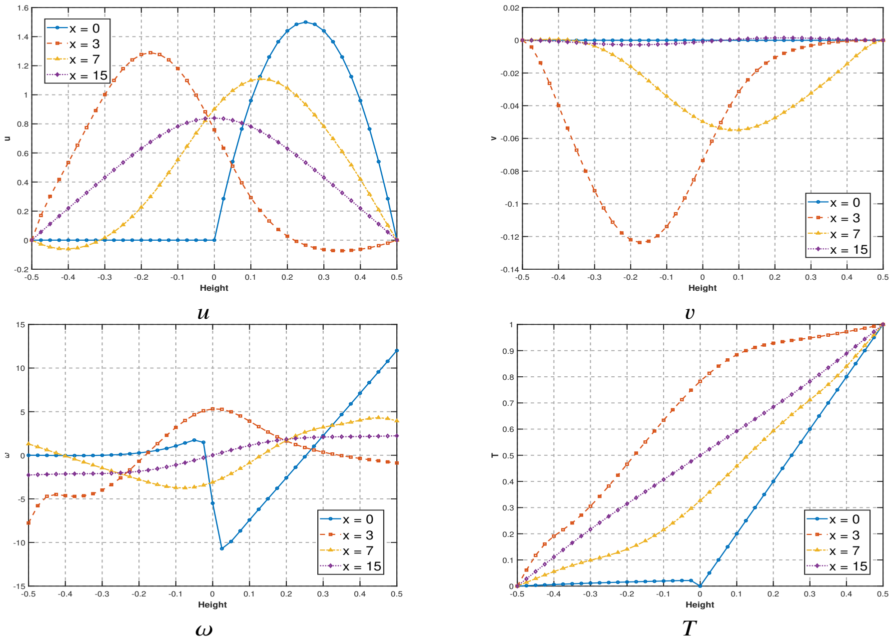


Fig. 2 Vertical profiles of u, v, ω, and T at $x = 0, 3, 7$, and 15.

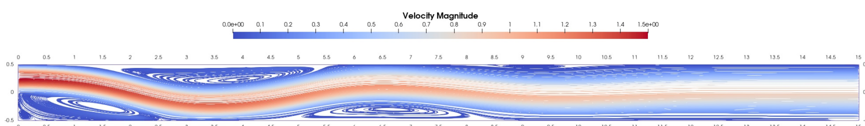


Fig. 3 Streamline contours for the backward-facing step flow at $Re = 800$.

For $Re = 800$, Fig. 2 shows the vertical profiles of u, v, ω and T at $x = 0, 3, 7$, and 15, while Fig. 3 depicts the corresponding streamlines in an extended domain ($0 < x < 15$). The flow descends from the inlet, rebounds, and slightly descends again before approaching the stratified Poiseuille solution. The structure is characterized by three bottom-wall and two top-wall eddies, consistent with the findings in [13].

We run on a cluster of two 24-core Intel Xeon Platinum CPUs. Fig. 4 compares the convergence history of the Newton residuals using the INB and Right-PIN methods across different Reynolds numbers, respectively. The results demonstrate the superior robustness and efficiency of the Right-PIN approach. In contrast, INB fails to converge for $Re > 600$ due to line search failures. Table 1 examines the impact of the relative tolerance $\epsilon_{\text{global-linear-rtol}}$ for the global Jacobian systems on the performance of Right-PIN. Relaxing the tolerance $\epsilon_{\text{global-linear-rtol}}$ from 10^{-6} to 10^{-3} often results in a comparable or slightly higher number of Newton iterations, yet it consistently and significantly reduces the total compute time, which is particularly evident at higher Reynolds numbers on the finer mesh, despite occasional increases in compute time in a few instances.

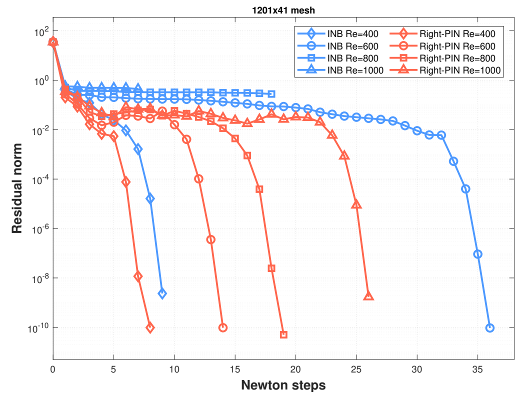


Fig. 4 The convergence history of the Newton residuals for the backward-facing step flow using INB and Right-PIN on the 1201 × 41 mesh, respectively.

Table 1 Comparison of Newton iterations and total compute time (in seconds) for Right-PIN under different global linear relative tolerances $\epsilon_{\text{global-linear-rtol}}$. All simulations were performed using 10 (10 × 1) processors.

	$\epsilon_{\text{global-linear-rtol}} = 10^{-6}$				$\epsilon_{\text{global-linear-rtol}} = 10^{-3}$			
Mesh size	1201 × 41		1801 × 61		1201 × 41		1801 × 61	
Re	Newton	Time	Newton	Time	Newton	Time	Newton	Time
100	4	40.8	4	129.1	4	40.5	4	126.8
200	5	52.7	5	162.6	5	56.9	5	187.0
300	6	69.0	7	252.4	7	71.3	7	220.5
400	8	86.9	9	357.3	8	82.1	9	294.2
500	14	171.5	14	554.8	17	172.2	14	434.1
600	14	158.1	15	583.2	16	160.5	16	507.6
700	17	205.2	27	1093.7	18	181.4	22	688.3
800	19	223.6	22	890.1	15	178.6	19	600.2
900	16	227.1	26	1249.6	24	245.9	24	762.7
1000	26	382.3	24	1184.5	22	223.7	27	851.6

5 Conclusions

For the stratified backward-facing step flow problem, our results establish the effectiveness of the right-side Oseen-like nonlinear preconditioning strategy. Numerical experiments demonstrate that the proposed Right-PIN method offers superior robustness over the standard approach. These findings provide a foundation for developing effective solvers for other challenging problems in computational fluid dynamics, with natural extensions to three-dimensional configurations in future work.

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References

1. Balay, S., Abhyankar, S., Adams, M.F., Benson, S., Brown, J., Brune, P., Buschelman, K., Constantinescu, E.M., Dalcin, L., A., D., Eijkhout, V., Faibussowitsch, J., Gropp, W.D., Hapla, V., Isaac, T., Jolivet, P., D., K., Kaushik, D., Knepley, M.G., Kong, F., Kruger, S., A., M.D., McInnes, L.C., T., M.R., Mitchell, L., T., M., Roman, J.E., Rupp, K., P., S., Sarich, J., Smith, B.F., Zampini, S., Zhang, H., Zhang, H., Zhang, J.: PETSc Web page. https://petsc.org/ (2025). URL https://petsc.org/
2. Batchelor, G.: An Introduction to Fluid Dynamics. Cambridge (2000). Second ed.
3. Cai, X.C., Keyes, D.E.: Nonlinearly preconditioned inexact Newton algorithms. SIAM J. Sci. Comput. **24**(1), 183–200 (2002)
4. Cai, X.C., Li, X.F.: Inexact Newton methods with restricted additive Schwarz based nonlinear elimination for problems with high local nonlinearity. SIAM J. Sci. Comput. **33**(2), 746–762 (2011)
5. Cai, X.C., Sarkis, M.: A restricted additive Schwarz preconditioner for general sparse linear systems. SIAM J. Sci. Comput. **21**(2), 792–797 (1999)
6. Coffey, T.S., Kelley, C.T., Keyes, D.E.: Pseudotransient continuation and differential-algebraic equations. SIAM J. Sci. Comput. **25**(2), 553–569 (2003)
7. Dolean, V., Gander, M.J., Kwok, F., Masson, R., Kheriji, W.: Nonlinear preconditioning: How to use a nonlinear Schwarz method to precondition Newton's method. SIAM J. Sci. Comput. **38**(6), 3357–3380 (2016)
8. Erturk, E.: Numerical solutions of 2-D steady incompressible flow over a backward-facing step, Part I: High Reynolds number solutions. Comput. Fluids **37**(6), 633–655 (2008)
9. Heinlein, A., Lanser, M.: Additive and hybrid nonlinear two-level Schwarz methods and energy minimizing coarse spaces for unstructured grids. SIAM J. Sci. Comput. **42**(4), A2461–A2488 (2020)
10. Klawonn, A., Lanser, M., Rheinbach, O.: Nonlinear FETI-DP and BDDC methods. SIAM J. Sci. Comput. **36**(2), A737–A765 (2014)
11. Knoll, D.A., Keyes, D.E.: Jacobian-free Newton–Krylov methods: a survey of approaches and applications. J. Comput. Phys. **193**(2), 357–397 (2004)
12. Lanzkron, P.J., Rose, D.J., Wilkes, J.T.: An analysis of approximate nonlinear elimination. SIAM J. Sci. Comput. **17**(2), 538–559 (1996)
13. Leone Jr, J.M.: Open boundary condition symposium benchmark solution: Stratified flow over a backward-facing step. Int. J. Numer. Meth. Fluids **11**(7), 969–984 (1990)
14. Liu, L., Gao, W., Keyes, D.E.: Comparison of Jacobian-preserving nonlinear preconditioning techniques. SIAM J. Sci. Comput. (in review) (2025)
15. Liu, L., Keyes, D.E.: Field-split preconditioned inexact Newton algorithms. SIAM J. Sci. Comput. **37**(3), 1388–1409 (2015)
16. Prudencio, E.E., Byrd, R., Cai, X.C.: Parallel full space SQP Lagrange–Newton–Krylov–Schwarz algorithms for PDE-constrained optimization problems. SIAM J. Sci. Comput. **27**(4), 1305–1328 (2006)
17. Smooke, M.D., Mattheij, R.M.: On the solution of nonlinear two-point boundary value problems on successively refined grids. Appl. Numer. Math. **1**(6), 463–487 (1985)
18. Yang, H.J., Hwang, F.N.: An adaptive nonlinear elimination preconditioned inexact Newton algorithm for highly local nonlinear multicomponent PDE systems. Appl. Numer. Math. **133**(nov.), 100–115 (2018)
19. Yang, H.J., Hwang, F.N., Cai, X.C.: Nonlinear preconditioning techniques for full-space Lagrange–Newton solution of PDE-constrained optimization problems. SIAM J. Sci. Comput. **38**(5), A2756–A2778 (2016)
20. Yang, H.J., Yang, C., Sun, S.Y.: Active-set reduced-space methods with nonlinear elimination for two-phase flow problems in porous media. SIAM J. Sci. Comput. **38**(4), B593–B618 (2016)