Coarse Spaces Using Extended Generalized Eigenproblems for Heterogeneous Helmholtz Problems

Emile Parolin[0000−0002−6512−7980],
Frédéric Nataf[0000−0003−2813−3481]

1 Introduction

An abstract construction of coarse spaces for non-Hermitian problems and non-Hermitian domain decomposition preconditioners based on extended generalized eigenproblems was proposed in [7] and analyzed on the matrix formulation. Building upon this work, we consider instead here the specific case of heterogeneous Helmholtz problems, and the derivation and analysis is performed at the continuous level. Albeit different from its derivation, its use of oversampling and the underlying eigenproblems, our approach shares similarities with the methods in [4, 6].

2 Optimized Restricted Additive Schwarz method

Model problem. Let $d \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. We consider the heterogeneous Helmholtz equation for $\omega > 0$

$$\text{Find } u \in H^1(\Omega) \text{ such that:} \qquad \begin{cases} -\nabla \cdot (\mu \nabla u) - \omega^2 \nu u = f, & \text{in } \Omega, \\ \mu \partial_{\mathbf{n}} u - \imath \omega u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where μ and ν are two variable but positive and bounded coefficients, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ model the source terms, and $\mathbf{n}$ is the outward unit normal vector to the boundary $\partial\Omega$. The model problem (1) takes the following variational form

---

Emile Parolin
Sorbonne Université, Université Paris Cité, CNRS, INRIA, Laboratoire Jacques-Louis Lions, LJLL, EPC ALPINES, F-75005 Paris, France, e-mail: emile.parolin@inria.fr

Frédéric Nataf
Sorbonne Université, Université Paris Cité, CNRS, INRIA, Laboratoire Jacques-Louis Lions, LJLL, EPC ALPINES, F-75005 Paris, France, e-mail: frederic.nataf@sorbonne-universite.fr

$$\text{Find } u \in H^1(\Omega) \text{ such that:} \qquad a_\Omega(u, u^t) = l_\Omega(u^t), \quad \forall u^t \in H^1(\Omega), \tag{2}$$

where we introduced, for any $D \subset \Omega$ and any $u, u^t \in H^1(D)$:

$$a_D(u, u^t) := (\mu \nabla u, \nabla u^t)_{L^2(D)^d} - \omega^2(\nu u, u^t)_{L^2(D)} - \imath\omega(u, u^t)_{L^2(\partial D \cap \partial\Omega)}, \tag{3}$$

$$l_D(u^t) := (f, u^t)_{L^2(D)} + (g, u^t)_{L^2(\partial D \cap \partial\Omega)}. \tag{4}$$

The above model problem (2) is well-posed using standard arguments based on the Fredholm alternative and a unique continuation principle.

One-level domain decomposition algorithm. We introduce a decomposition of Ω into $J \in \mathbb{N}$, $J > 0$, overlapping open subdomains Ω_j, $j = 1, \ldots, J$, associated to a partition-of-unity function $\chi_j \in W^{1,\infty}(\Omega_j)$ such that

$$\overline{\Omega} = \textstyle\bigcup_j \overline{\Omega_j}, \qquad \chi_j \geq 0,\ \chi_j|_{\partial\Omega_j \setminus \partial\Omega} = 0, \qquad \textstyle\sum_j E_j \chi_j u|_{\Omega_j} = u,\ \forall u \in H^1(\Omega), \tag{5}$$

where E_j is an extension-by-continuity operator. More precisely, we require that $E_j : H^1(\Omega_j) \to H^1(\Omega)$ continuously, $E_j u_j|_{\Omega_j} = u_j$ and $\operatorname{supp}(E_j u_j) \subset \widetilde{\Omega}_j$ for any $u_j \in H^1(\Omega_j)$ and for some extended open subdomain $\widetilde{\Omega}_j$ such that $\Omega_j \subsetneq \widetilde{\Omega}_j \subset \Omega$.

Starting from an initial guess $u_{\mathrm{I}}^0 \in H^1(\Omega)$, the one-level (the index I stands for one-level) Optimized Restricted Additive Schwarz (ORAS) algorithm defines the sequence

$$u_{\mathrm{I}}^{n+1} := u_{\mathrm{I}}^n + \textstyle\sum_j E_j \chi_j v_j^{n+1}, \qquad n \geq 0, \tag{6}$$

where v_j^{n+1} is solution of: Find $v_j^{n+1} \in H^1(\Omega_j)$ such that:

$$b_{\Omega_j}(v_j^{n+1}, v_j^t) = l_\Omega(E_j v_j^t) - a_\Omega(u_{\mathrm{I}}^n, E_j v_j^t), \qquad \forall v_j^t \in H^1(\Omega_j), \tag{7}$$

where we introduced, for any $D \subset \Omega$ and any $v, v^t \in H^1(D)$:

$$b_D(v, v^t) := a_D(v, v^t) - \imath\omega(v, v^t)_{L^2(\partial D \setminus \partial\Omega)}. \tag{8}$$

By similar arguments as for the model problem, the local problem (7) is well-posed, and we remark that the right-hand-side in (7) can be computed locally in $\widetilde{\Omega}_j$ so that v_j^{n+1} is equivalently solution of

$$b_{\Omega_j}(v_j^{n+1}, v_j^t) = l_{\widetilde{\Omega}_j}((E_j v_j^t)|_{\widetilde{\Omega}_j}) - a_{\widetilde{\Omega}_j}(u_{\mathrm{I}}^n|_{\widetilde{\Omega}_j}, (E_j v_j^t)|_{\widetilde{\Omega}_j}), \qquad \forall v_j^t \in H^1(\Omega_j). \tag{9}$$

For general partitions (where at least three subdomains overlap in some part of the domain Ω), the sequence of u_{I}^n depends a priori on the choice of E_j (which is not unique), since it affects the right-hand-side of (7) (in (6), the presence of the partition-of-unity χ_j implies that E_j extends by zero, which is uniquely defined).

One-level error analysis. For $n \geq 0$, the one-level error $e_{\mathrm{I}}^n := u - u_{\mathrm{I}}^n$ satisfies

$$e_{\mathrm{I}}^{n+1} = u - u_{\mathrm{I}}^{n+1} = e_{\mathrm{I}}^n - \textstyle\sum_j E_j \chi_j v_j^{n+1} = \textstyle\sum_j E_j \chi_j (e_{\mathrm{I}}^n|_{\Omega_j} - v_j^{n+1}). \tag{10}$$

From (9), we have,

$$b_{\Omega_j}(e_{\mathrm{I}}^n|_{\Omega_j} - v_j^{n+1}, v_j^t) = b_{\Omega_j}(e_{\mathrm{I}}^n|_{\Omega_j}, v_j^t) - a_{\widetilde{\Omega}_j}(e_{\mathrm{I}}^n|_{\widetilde{\Omega}_j}, (E_j v_j^t)|_{\widetilde{\Omega}_j}),\ \forall v_j^t \in H^1(\Omega_j), \tag{11}$$

so that

$$e_{\mathrm{I}}^{n+1} = \textstyle\sum_j E_j \chi_j R_j e_{\mathrm{I}}^n|_{\widetilde{\Omega}_j}, \tag{12}$$

where $R_j : H^1(\widetilde{\Omega}_j) \to H^1(\Omega_j)$ is defined for any $\widetilde{v}_j \in H^1(\widetilde{\Omega}_j)$ as

$$b_{\Omega_j}(R_j \widetilde{v}_j, v_j^t) = b_{\Omega_j}(\widetilde{v}_j|_{\Omega_j}, v_j^t) - a_{\widetilde{\Omega}_j}(\widetilde{v}_j, (E_j v_j^t)|_{\widetilde{\Omega}_j}), \qquad \forall v_j^t \in H^1(\Omega_j). \tag{13}$$

The operator R_j is well-defined since the local problems (13) are well-posed.

We measure the error in the energy norm given by the following sesquilinear form: for any $D \subset \Omega$ and any $u, u^t \in H^1(D)$, let

$$c_D(u, u^t) := (\mu \nabla u, \nabla u^t)_{L^2(D)^d} + \omega^2 (\nu u, u^t)_{L^2(D)}, \qquad \|u\|_D^2 := c_D(u, u). \tag{14}$$

The error can be estimated as follows

$$\|e_{\mathrm{I}}^{n+1}\|_{\Omega}^2 \le k_0 \textstyle\sum_j \left\|\chi_j R_j e_{\mathrm{I}}^n|_{\widetilde{\Omega}_j}\right\|_{\Omega_j}^2 \le k_0 \xi \textstyle\sum_j \left\|e_{\mathrm{I}}^n|_{\widetilde{\Omega}_j}\right\|_{\widetilde{\Omega}_j}^2 \le k_0 k_1 \xi \|e_{\mathrm{I}}^n\|_{\Omega}^2, \tag{15}$$

where we introduced

$$k_0 := \sup_{\substack{(u_j)_j \\ 0 \ne u_j \in H_0^1(\Omega_j)}} \frac{\|\sum_j E_j u_j\|_{\Omega}^2}{\sum_j \|u_j\|_{\Omega_j}^2}, \qquad k_1 := \sup_{0 \ne u \in H^1(\Omega)} \frac{\sum_j \|u|_{\widetilde{\Omega}_j}\|_{\widetilde{\Omega}_j}^2}{\|u\|_{\Omega}^2}, \tag{16}$$

$$\xi := \sup_j \xi_j, \qquad \xi_j := \sup_{0 \ne \widetilde{u}_j \in H^1(\widetilde{\Omega}_j)} \frac{\|\chi_j R_j \widetilde{u}_j\|_{\Omega_j}^2}{\|\widetilde{u}_j\|_{\widetilde{\Omega}_j}^2}. \tag{17}$$

The estimate (15) shows that a sufficient condition to have convergence of the algorithm is that $k_0 k_1 \xi < 1$, which is unlikely to be satisfied. The two constants k_0 and k_1 can be bounded uniformly of the parameters μ, ν, ω of the problem and only depend respectively on the maximum number of neighbors for a subdomain (see e.g. [2, Lem. 7.9]) and the maximum number of subdomain intersection multiplicity (see e.g. [2, Lem. 7.13]). For reasonable partitions, both quantities can be bounded uniformly of the number of subdomains. On the other hand, ξ depends on the parameters of the problem. In the next section, we propose the construction of a coarse space and associated two-level domain decomposition algorithm that allows to control each ξ_j independently, hence ξ.

3 Two-level ORAS method

Local generalized eigenproblem The definition of ξ_j suggests considering the following local generalized eigenproblem: Find $(\widetilde{u}_j, \lambda_j) \in H^1(\widetilde{\Omega}_j) \times \mathbb{R}$ such that:

$$c_{\Omega_j}(\chi_j R_j \widetilde{u}_j, \chi_j R_j \widetilde{u}^t_j) = \lambda_j c_{\widetilde{\Omega}_j}(\widetilde{u}_j, \widetilde{u}^t_j), \qquad \forall \widetilde{u}^t_j \in H^1(\widetilde{\Omega}_j). \tag{18}$$

This problem is equivalent to the following saddle-point formulation:
Find $(\widetilde{u}_j, v_j, \sigma_j, \lambda_j) \in H^1(\widetilde{\Omega}_j) \times H^1(\Omega_j) \times H^1(\Omega_j) \times \mathbb{R}$ such that:

$$\begin{cases} c_{\Omega_j}(\chi_j v_j, \chi_j v^t_j) - b_{\Omega_j}(v^t_j, \sigma_j) = 0, & \forall v^t_j \in H^1(\Omega_j), \\ -b_{\Omega_j}(v_j, \sigma^t_j) + b_{\Omega_j}(\widetilde{u}_j|_{\Omega_j}, \sigma^t_j) - a_{\widetilde{\Omega}_j}(\widetilde{u}_j, (E_j \sigma^t_j)|_{\widetilde{\Omega}_j}) = 0, & \forall \sigma^t_j \in H^1(\Omega_j), \\ b_{\Omega_j}(\widetilde{u}^t_j|_{\Omega_j}, \sigma_j) - a_{\widetilde{\Omega}_j}(\widetilde{u}^t_j, (E_j \sigma_j)|_{\widetilde{\Omega}_j}) = \lambda_j c_{\widetilde{\Omega}_j}(\widetilde{u}_j, \widetilde{u}^t_j), & \forall \widetilde{u}^t_j \in H^1(\widetilde{\Omega}_j). \end{cases} \tag{19}$$

The following result shows that each ξ_j (hence ξ) is finite and that the eigenvalues λ_j solutions to (18) converge to 0.

Proposition 1 *The operator* $K_j : \widetilde{u}_j \in H^1(\widetilde{\Omega}_j) \mapsto \chi_j R_j \widetilde{u}_j \in H^1(\Omega_j)$ *is compact.*

Proof. The result is based on a Caccioppoli inequality which we establish next (see also [5, Lem. 2.7]). From (13) we have, for any $\widetilde{u}_j \in H^1(\widetilde{\Omega}_j)$,

$$b_{\Omega_j}(R_j \widetilde{u}_j, \chi_j^2 R_j \widetilde{u}_j) = 0, \tag{20}$$

which implies (adding (20) and its conjugate) that

$$\begin{aligned} &(\mu \chi_j \nabla R_j \widetilde{u}_j, \chi_j \nabla R_j \widetilde{u}_j)_{L^2(\Omega_j)^d} + (\mu R_j \widetilde{u}_j \nabla \chi_j, \chi_j \nabla R_j \widetilde{u}_j)_{L^2(\Omega_j)^d} \\ &\quad + (\mu \chi_j \nabla R_j \widetilde{u}_j, R_j \widetilde{u}_j \nabla \chi_j)_{L^2(\Omega_j)^d} - \omega^2 (\nu R_j \widetilde{u}_j, \chi_j^2 R_j \widetilde{u}_j)_{L^2(\Omega_j)} = 0. \end{aligned} \tag{21}$$

On the other hand, we have, for any $\widetilde{u}_j \in H^1(\widetilde{\Omega}_j)$,

$$\begin{aligned} \|\chi_j R_j \widetilde{u}_j\|^2_{\Omega_j} &= c_{\Omega_j}(\chi_j R_j \widetilde{u}_j, \chi_j R_j \widetilde{u}_j) = (\mu R_j \widetilde{u}_j \nabla \chi_j, R_j \widetilde{u}_j \nabla \chi_j)_{L^2(\Omega_j)^d} \\ &\quad + (\mu \chi_j \nabla R_j \widetilde{u}_j, \chi_j \nabla R_j \widetilde{u}_j)_{L^2(\Omega_j)^d} + (\mu R_j \widetilde{u}_j \nabla \chi_j, \chi_j \nabla R_j \widetilde{u}_j)_{L^2(\Omega_j)^d} \\ &\quad + (\mu \chi_j \nabla R_j \widetilde{u}_j, R_j \widetilde{u}_j \nabla \chi_j)_{L^2(\Omega_j)^d} + \omega^2 (\nu \chi_j R_j \widetilde{u}_j, \chi_j R_j \widetilde{u}_j)_{L^2(\Omega_j)} \\ &= (\mu R_j \widetilde{u}_j \nabla \chi_j, R_j \widetilde{u}_j \nabla \chi_j)_{L^2(\Omega_j)^d} + 2\omega^2 (\nu R_j \widetilde{u}_j, \chi_j^2 R_j \widetilde{u}_j)_{L^2(\Omega_j)} \leq C \|R_j \widetilde{u}_j\|^2_{L^2(\Omega_j)}, \end{aligned}$$

for some constant $C > 0$ depending on ω, upper bounds on the coefficients μ, ν and the partition-of-unity χ_j. The result then follows by application of the Rellich–Kondrachov theorem. □

Coarse space. For a user-defined parameter $\tau > 0$, we introduce the coarse space

$$U_\tau := \bigoplus_j \operatorname{span}\{E_j \chi_j R_j \widetilde{u}_j \ : \ (\widetilde{u}_j, \lambda_j) \text{ solution of (18) such that } \lambda_j > \tau\}. \tag{22}$$

The coarse space U_τ is made of local Helmholtz solutions $R_j\widetilde{u}_j$ that are extended-by-zero globally thanks to E_j after being multiplied by the partition-of-unity function χ_j. Indeed, for $\widetilde{u}_j$ solution of (18), we have $R_j\widetilde{u}_j = v_j$ solution of (19) which satisfies (testing with $\sigma_j^t \in H_0^1(\Omega_j)$ in (19))

$$(\mu\nabla v_j, \nabla v_j^t)_{L^2(\Omega_j)^d} - \omega^2(\nu v_j, v_j^t)_{L^2(\Omega_j)} = 0, \qquad \forall v_j^t \in H_0^1(\Omega_j). \tag{23}$$

Moreover, thanks to the partition-of-unity, the coarse space is conforming, namely $U_\tau \subset H^1(\Omega)$. Finally, Proposition 1 ensures that U_τ is finite dimensional.

Two-level domain decomposition algorithm. Starting from an initial guess $u_{\mathrm{II}}^0 \in H^1(\Omega)$, we define a sequence of solutions $u_{\mathrm{II}}^n \in H^1(\Omega)$ (the index $_{\mathrm{II}}$ stands for two-level),

$$u_{\mathrm{II}}^{n+1} := u_{\mathrm{II}}^n + \textstyle\sum_j E_j\chi_j w_j^{n+1} + w_0^{n+1}, \qquad n \geq 0, \tag{24}$$

where w_j^{n+1} is solution of: Find $w_j^{n+1} \in H^1(\Omega_j)$ such that:

$$b_{\Omega_j}(w_j^{n+1}, w_j^t) = l_\Omega(E_j w_j^t) - a_\Omega(u_{\mathrm{II}}^n, E_j w_j^t), \qquad \forall w_j^t \in H^1(\Omega_j), \tag{25}$$

and w_0^{n+1} is solution of: Find $w_0^{n+1} \in U_\tau$ such that:

$$a_\Omega(w_0^{n+1}, w_0^t) = l_\Omega(w_0^t) - a_\Omega(u_{\mathrm{II}}^n + \textstyle\sum_j E_j\chi_j w_j^{n+1}, w_0^t), \quad \forall w_0^t \in U_\tau. \tag{26}$$

Assumption 1. *For any anti-linear form L on* $H^1(\Omega)$*, consider the two problems*

$$\text{Find } u \in H^1(\Omega) \text{ such that:} \quad a_\Omega(u, u^t) = L(u^t),\ \forall u^t \in H^1(\Omega), \tag{27}$$

$$\text{Find } u_0 \in U_\tau \text{ such that:} \quad a_\Omega(u_0, u_0^t) = L(u_0^t),\ \forall u_0^t \in U_\tau. \tag{28}$$

In the following, we assume that the coarse problem (28) *is well-posed and that there exists* $\sigma > 0$*, independent of* τ *and L, such that the following stability bound holds*

$$\|u - u_0\|_\Omega \leq \sigma\|u\|_\Omega. \tag{29}$$

Assumption 1 implies that the problem (26) is well-posed. The validity of Assumption 1 can be argued for large enough coarse spaces U_τ (i.e. taking the user-controlled parameter τ sufficiently small). For instance, well-posedness of the coarse problem and a quasi-optimality estimate implying (29) can be obtained using the so-called Aubin–Nitsche duality trick and Schatz argument, see [3, Lemma 5.1].

Two-level error analysis. For $n \geq 0$, the two-level error $e_{\mathrm{II}}^n := u - u_{\mathrm{II}}^n$ satisfies

$$e_{\mathrm{II}}^{n+1} = u - u_{\mathrm{II}}^{n+1} = u - u_{\mathrm{II}}^n - \textstyle\sum_j E_j\chi_j w_j^{n+1} - w_0^{n+1}. \tag{30}$$

From (25), we have,

$$b_{\Omega_j}(e_{\mathrm{II}}^n|_{\Omega_j} - w_j^{n+1}, w_j^t) = b_{\Omega_j}(e_{\mathrm{II}}^n|_{\Omega_j}, w_j^t) - a_{\widetilde{\Omega}_j}(e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}, (E_j w_j^t)|_{\widetilde{\Omega}_j}),\ \forall w_j^t \in H^1(\Omega_j), \tag{31}$$

and from (26), we have

$$a_\Omega(w_0^{n+1}, w_0^t) = a_\Omega\big(\textstyle\sum_j E_j\chi_j(e_{\mathrm{II}}^n|_{\Omega_j} - w_j^{n+1}), w_0^t\big), \quad \forall w_0^t \in U_\tau, \tag{32}$$

so that

$$e_{\mathrm{II}}^{n+1} = (\mathrm{I} - P_0)\textstyle\sum_j E_j\chi_j R_j e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}, \tag{33}$$

where we introduced $P_0 : H^1(\Omega) \to H^1(\Omega)$ defined as: for any $u \in H^1(\Omega)$, $P_0 u \in U_\tau$ such that:

$$a_\Omega(P_0 u, u_0^t) = a_\Omega(u, u_0^t), \quad \forall u_0^t \in U_\tau. \tag{34}$$

Provided Assumption 1 holds, the operator P_0 is well-defined. Besides, P_0 is an oblique projection on the closed space U_τ and the operator norm of $\mathrm{I} - P_0$ for c_Ω is bounded by the stability constant σ (see (29)) of the coarse problem, namely

$$\sup_{0 \neq u \in H^1(\Omega)} \frac{\|(\mathrm{I} - P_0)u\|_\Omega}{\|u\|_\Omega} \leq \sigma. \tag{35}$$

We can now state our main result, which is the counterpart for the heterogeneous Helmholtz problem of the matrix formulation result stated in [7, Th. 2.12].

Theorem 1 *If the user-controlled parameter τ is taken to be sufficiently small such that Assumption 1 holds and $\rho := \sigma\sqrt{k_0 k_1 \tau} < 1$, then the iterative solution u_{II}^n defined in* (24) *converges towards the exact solution u of the model problem* (2)*, in the norm given by c_Ω defined in* (14)*, and with rate ρ, namely*

$$\|u - u_{\mathrm{II}}^n\|_\Omega \leq \rho^n \|u - u_{\mathrm{II}}^0\|_\Omega, \qquad n \geq 0. \tag{36}$$

Proof. Let Π_j be the $c_{\widetilde{\Omega}_j}$-orthogonal projection on the finite dimensional space

$$\mathrm{span}\{\widetilde{u}_j \ : \ (\widetilde{u}_j, \lambda_j) \text{ solution of (18) such that } \lambda_j > \tau\}. \tag{37}$$

Then, by construction of the coarse space, the image of $\sum_j E_j\chi_j R_j \Pi_j$ is in the kernel of $\mathrm{I} - P_0$, hence

$$e_{\mathrm{II}}^{n+1} = (\mathrm{I}-P_0)\textstyle\sum_j E_j\chi_j R_j[\Pi_j + (\mathrm{I}-\Pi_j)]e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j} = (\mathrm{I}-P_0)\textstyle\sum_j E_j\chi_j R_j(\mathrm{I}-\Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}, \tag{38}$$

and moreover (see e.g. [2, Lem. 7.7])

$$\|\chi_j R_j(\mathrm{I} - \Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\|_{\Omega_j}^2 \leq \tau\|(\mathrm{I} - \Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\|_{\widetilde{\Omega}_j}^2 \leq \tau\|e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\|_{\widetilde{\Omega}_j}^2. \tag{39}$$

Recalling (16) and (35), the error can hence be estimated as follows

$$\begin{aligned}
\|e_{\mathrm{II}}^{n+1}\|_\Omega^2 &= \big\|(\mathrm{I} - P_0)\textstyle\sum_j E_j\chi_j R_j(\mathrm{I} - \Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\big\|_\Omega^2 \\
&\leq \sigma^2 \,\big\|\textstyle\sum_j E_j\chi_j R_j(\mathrm{I} - \Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\big\|_\Omega^2 \leq \sigma^2 k_0 \ \textstyle\sum_j \big\|\chi_j R_j(\mathrm{I} - \Pi_j)e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\big\|_{\Omega_j}^2 \\
&\leq \sigma^2 k_0 \tau \ \textstyle\sum_j \big\|e_{\mathrm{II}}^n|_{\widetilde{\Omega}_j}\big\|_{\widetilde{\Omega}_j}^2 \leq \sigma^2 k_0 k_1 \tau \ \|e_{\mathrm{II}}^n\|_\Omega^2.
\end{aligned} \tag{40}$$

□

In practice, Krylov acceleration is used, and one iteration of the above algorithm (24) is used as a preconditioner. Theorem 1 also implies convergence of the Krylov iterative method with this preconditioner.

4 Numerical illustrations

We conclude with some numerical experiments on the Marmousi model, which represents a realistic geological structure with varying velocity c (see Figure 1) and fits within the model problem (1) with $\nu = c^{-2}$, $\mu = 1$. A homogeneous Dirichlet boundary condition is applied on the surface and Robin boundary conditions are imposed on the remaining boundaries. The source term f is a regularized Dirac mass (sharp Gaussian function) centered near the surface (and $g = 0$).

The implementation is performed in FreeFEM using the FFDDM framework. The discretization consists of $\mathbb{P}_2$ Lagrange finite elements. We perform a weak scaling test using a frequency ramp, so that the number of degrees of freedom in each subdomain remains constant around ~ 36000 (~ 780 on the subdomain boundary). We compare results for the one-level ORAS preconditioner and the two-level preconditioners for three coarse space sizes. In practice, instead of using the parameter τ to filter the eigenvalues, we compute a fixed number of eigenpairs per subdomain given as a percentage of the number of degrees of freedom on the boundary of the subdomain (which is an upper bound on the meaningful eigenvectors to include on the coarse space). This strategy avoids computing too many unnecessary eigenpairs and allows more sensible run time comparisons. The numerical results are reported in Table 1 and in Figure 2.

The results show the poor scalability of the one-level preconditioner for increasing ω. For a large enough coarse space, the second-level preconditioner converges in a much reduced number of iterations, albeit still growing with respect to ω. This is however achieved at the expense of a larger setup time (including factorization of local matrices, computation of eigenvalues, assembly and factorization of the coarse problem). At some point, the reduction in iterations obtained by increasing further the coarse space is offset by the larger computational effort for building the coarse space and solving the coarse problem.

Additional numerical results providing a numerical comparison with alternative methods (including the coarse spaces in [4, 6]) are given in [1].

References

1. Dolean, V., Fry, M., Langer, M., Parolin, E., Tournier, P.H.: Achieving wavenumber robustness in domain decomposition for heterogeneous helmholtz equations: an overview of spectral coarse spaces. arXiv preprint arXiv:2509.02131 (2025)
2. Dolean, V., Jolivet, P., Nataf, F.: An introduction to domain decomposition methods. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2015). Algorithms, theory, and parallel implementation
3. Galkowski, J., Spence, E.A.: Convergence theory for two-level hybrid schwarz preconditioners for high-frequency helmholtz problems. arXiv preprint arXiv:2501.11060 (2025)

			one-level			two-level (7.5%)				two-level (10%)				two-level (12.5%)			
ω	J	N	It	T_{setup}	T_{solver}	CS	It	T_{setup}	T_{solver}	CS	It	T_{setup}	T_{solver}	CS	It	T_{setup}	T_{solver}
30π	32	1106441	224	1.6	37	1884	25	55	5.4	2512	16	79	4.1	3140	13	105	3.9
42π	64	2167257	314	1.7	62	3732	46	55	9.3	4976	20	80	5.0	6220	15	107	4.2
60π	128	4420881	478	1.5	125	7524	94	58	20	10032	27	84	6.7	12540	20	113	5.7
84π	256	8662193	645	1.7	267	14865	202	67	70	19820	42	96	17	24775	26	134	14
120π	512	17673761	915	2.2	1120	30297	389	74	178	40396	62	109	28	50495	37	150	21

Table 1 Angular frequency ω, number of subdomains J, number of degrees of freedom N, number of GMRES iterations to reach a relative residual of 10^{-8} (It), setup time (T_{setup}) and solver time (T_{solver}) for the one-level preconditioner and two-level preconditioners for three coarse space sizes (CS) given as a percentage of the number of degrees of freedom on the skeleton (X%).

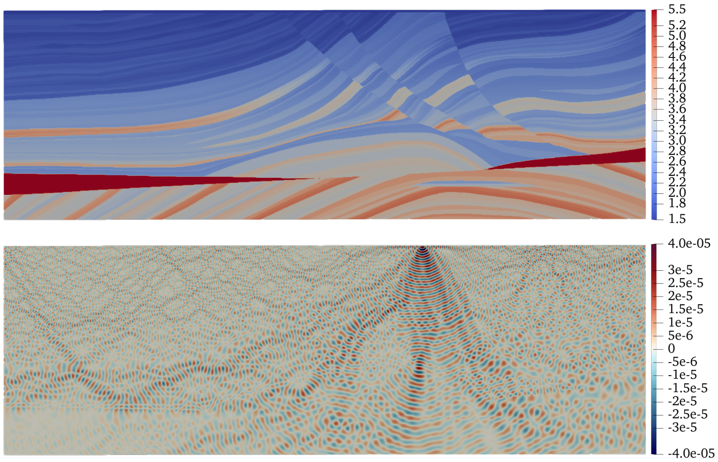

Fig. 1 Velocity profile (top) and real part of the solution u for $\omega = 60\pi$ (bottom).

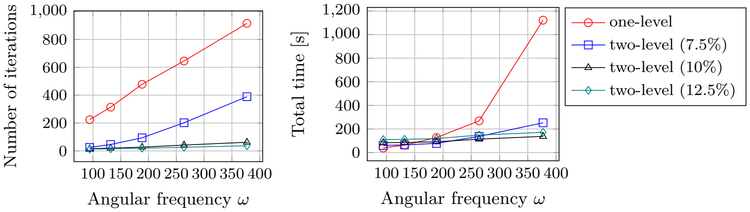

Fig. 2 GMRES iteration count (left) and total run time (right) with respect to ω.

4. Hu, Q., Li, Z.: A hybrid two-level weighted schwarz method for helmholtz equations. SIAM J. Numer. Anal. **63**(2), 716–743 (2025)
5. Ma, C., Alber, C., Scheichl, R.: Wavenumber-explicit convergence of a multiscale generalized finite element method for heterogeneous helmholtz problems. SIAM J. Numer. Anal. **61**(3), 1546–1584 (2023)
6. Ma, C., Alber, C., Scheichl, R., Zhang, Y.: Two-level restricted additive schwarz preconditioner based on multiscale spectral generalized fem for heterogeneous helmholtz problems. J. Sci. Comput. **105**(3), Paper No. 99 (2025)
7. Nataf, F., Parolin, E.: Coarse spaces for nonsymmetric two-level preconditioners based on local extended generalized eigenproblems. arXiv preprint arXiv:2404.02758 (2025)