Paving the Way to a T-Coercive Method for the Wave Equation

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1 Introduction

There has been an increased interest in finding space-time formulations for the wave equation that lead to unconditionally stable discretizations. Some of them rely on analyzing the exact mapping properties of the space-time wave operator in a variational setting [4, 5, 6, 9], including weak and least squares formulations. These often lead to a setting beyond standard Sobolev spaces, that is cumbersome to discretize. Others though, stay in the framework of Sobolev spaces, adding stabilization terms [3, 7], or applying special test functions [1, 2, 8]. The overall goal of all these approaches is to open the door to adaptivity in space and time simultaneously and without additional computational cost.

In this paper, we follow the idea from [8] and consider a second order ordinary differential equation (ODE) $u'' + \mu u = f$ with $\mu > 0$. This ODE can be linked to the wave equation and used to find a transformation operator such that one obtains a coercive and continuous bilinear form for the corresponding variational formulation. However, unlike previous approaches, we take pains to make sure the resulting transformation also depends on μ, and thus, when extended to the partial differential equation framework, can lead to a transformation that not only acts on time, but also on space. Moreover, we are interested in keeping standard Sobolev spaces as our solution spaces, such that the error analysis of future space-time methods and understanding of the solution is straightforward. With this purpose in mind, we deliberately insist on using the arising transformation to change the test space.

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Interestingly, this results in a variational formulation with a simple bilinear form, while only the right-hand side becomes more involved.

This contribution focuses on the key ideas for finding this transformation and therefore restricts attention to the ODE case. Ongoing and future work deals with the extension to the actual wave equation.

2 Motivation

Let $T > 0$, $\mu > 0$ and $f : (0,T) \to \mathbb{R}$ be given and let us consider the ordinary differential equation to find $u : [0,T] \to \mathbb{R}$ such that

$$u''(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \qquad u(0) = u'(0) = 0. \tag{1}$$

For the weak form, we multiply by a smooth function v and integrate by parts to get

$$\int_0^T \big[u''(t)v(t) + \mu u(t)v(t)\big]\, dt = \int_0^T \big[-u'(t)v'(t) + \mu u(t)v(t)\big]\, dt + u'(T)v(T).$$

Now, in order to get rid of the term $u'(T)$, about which we have no information, and to ensure well-defined integrals, we need the spaces

$$H^1_{0,}(0,T) := \{v \in H^1(0,T) |\, v(0) = 0\}, \qquad H^1_{,0}(0,T) := \{v \in H^1(0,T) |\, v(T) = 0\},$$

both endowed with the norm $\|v\|_{H^1_{0,}(0,T)} = \|v\|_{H^1_{,0}(0,T)} = \|v'\|_{L^2(0,T)}$. We consider for $f \in [H^1_{,0}(0,T)]'$, to find $u \in H^1_{0,}(0,T)$ such that for all $v \in H^1_{,0}(0,T)$

$$b_\mu(u,v) := \int_0^T \big[-u'(t)v'(t) + \mu u(t)v(t)\big]\, dt = \langle f, v\rangle_{[H^1_{,0}(0,T)]' \times H^1_{,0}(0,T)}. \tag{2}$$

Theorem 1 *[10, Lemma 4.2.1, Lemmata 4.2.3–4.2.4] The bilinear form $b_\mu : H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$ defined in* (2) *satisfies*

(B1) Boundedness: For all $u \in H^1_{0,}(0,T)$ and $v \in H^1_{,0}(0,T)$ it holds that

$$|b_\mu(u,v)| \le \left(1 + \frac{4T^2\mu}{\pi^2}\right) \|u'\|_{L^2(0,T)} \|v'\|_{L^2(0,T)}.$$

(B2) Bounded invertibility: For all $u \in H^1_{0,}(0,T)$ it holds that

$$\frac{2}{2 + T\sqrt{\mu}} \|u'\|_{L^2(0,T)} \le \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{b_\mu(u,v)}{\|v'\|_{L^2(0,T)}}.$$

(B3) Surjectivity: For all $0 \neq v \in H^1_{,0}(0,T)$ there exists $u_v \in H^1_{0,}(0,T)$ such that $b_\mu(u_v, v) \neq 0$.

The above theorem guarantees that the weak formulation (2) admits a unique solution $u \in H^1_{0,}(0,T)$ for all right-hand sides $f \in [H^1_{,0}(0,T)]'$ and fixed $\mu > 0$. But, the obtained bounds depend on $\mu > 0$ and, in particular, they degenerate when $\mu \to \infty$.

Our main contribution is to propose a formulation where we keep the solution $u \in H^1_{0,}(0,T)$ in the standard Sobolev space but get μ-independent bounds. For this, we introduce

$$\|w\|_{H^1_\mu} := \sqrt{\|w'\|^2_{L^2(0,T)} + \mu\|w\|^2_{L^2(0,T)}},$$

which resembles the space-time H^1-norm, and defines an equivalent norm on $H^1_{0,}(0,T)$ and $H^1_{,0}(0,T)$. In what follows, we will construct a transformation operator $\mathsf{T}_\mu : H^1_{0,}(0,T) \to H^1_{,0}(0,T)$ such that the bilinear form satisfies

$$\|u\|^2_{H^1_\mu} = b_\mu(u, \mathsf{T}_\mu\, u).$$

3 A space-time transformation!? Well, it's complex...

Let $\langle u, v\rangle := \int_0^T u(t)v(t)\,dt$, which for real-valued functions represents the L^2-inner product and the induced L^2-duality pairing. Now, using integration by parts, we get

$$\langle u', v\rangle + \langle u, v'\rangle = 0, \qquad \forall u \in H^1_{0,}(0,T), \forall v \in H^1_{,0}(0,T). \tag{3}$$

We introduce the operators

$$(\mathsf{D}^\pm_\mu; \mathrm{dom}(\mathsf{D}^\pm_\mu)) := (\pm i\partial_t + \sqrt{\mu}; H^1_{0,}(0,T)),\ \ (\widetilde{\mathsf{D}}^+_\mu; \mathrm{dom}(\widetilde{\mathsf{D}}^+_\mu)) := (i\partial_t + \sqrt{\mu}; H^1_{,0}(0,T)),$$

where i denotes the imaginary unit. Using (3), we have that

$$\langle \mathsf{D}^+_\mu\, u, \widetilde{\mathsf{D}}^+_\mu v\rangle = -\langle u', v'\rangle + \mu\langle u, v\rangle + i\sqrt{\mu}(\langle u', v\rangle + \langle u, v'\rangle) = b_\mu(u, v).$$

Moreover, note that for any real valued function $u \in H^1_{0,}(0,T)$

$$\begin{aligned}\langle \mathsf{D}^+_\mu\, u, \mathsf{D}^-_\mu\, u\rangle &= \langle \mathsf{D}^+_\mu\, u, \overline{\mathsf{D}^+_\mu\, u}\rangle = \left\|\Re(\mathsf{D}^+_\mu\, u)\right\|^2_{L^2(0,T)} + \left\|\Im(\mathsf{D}^+_\mu\, u)\right\|^2_{L^2(0,T)} \\ &= \mu\|u\|^2_{L^2(0,T)} + \|u'\|^2_{L^2(0,T)} = \|u\|^2_{H^1_\mu}.\end{aligned}$$

where, for any w, $\Re(w)$, $\Im(w)$ and $\overline{w}$ denote its real/imaginary part and complex conjugate, respectively. The above motivates us to introduce the operator

$$\mathcal{T}_\mu := \left(\widetilde{\mathsf{D}}^+_\mu\right)^{-1} \mathsf{D}^-_\mu, \tag{4}$$

since for $v = \mathcal{T}_\mu\, u$ we then formally have that

$$b_\mu(u, v) = \langle \mathsf{D}^+_\mu\, u, \widetilde{\mathsf{D}}^+_\mu\rangle\, \mathcal{T}_\mu\, u = \langle \mathsf{D}^+_\mu\, u, \mathsf{D}^-_\mu\, u\rangle = \|u\|^2_{H^1_\mu}.$$

3.1 The transformation operator: Its actual closed form formula

We proceed to compute the transformation operator defined in (4) acting on a function in $H^1_{0,}(0,T)$. Moreover, in order to stay consistent, we would like that the operator $\mathcal{T}_\mu$ maps functions from $H^1_{0,}(0,T)$ to functions with zero terminal condition.

As the solution of

$$\widetilde{\mathsf{D}^+_\mu} z := iz' + \sqrt{\mu} z = q \text{ in } (0,T), \quad z(T) = 0,$$

can be explicitly computed to be

$$z(t) = \left(\left(\widetilde{\mathsf{D}^+_\mu}\right)^{-1} q\right)(t) = i \int_t^T e^{i\sqrt{\mu}(t-s)} q(s)\, ds,$$

we have that for any $w \in H^1_{0,}(0,T)$

$$\begin{aligned} v(t) = (\mathcal{T}_\mu w)(t) &= \left(\left(\widetilde{\mathsf{D}^+_\mu}\right)^{-1} \mathsf{D}^-_\mu w\right)(t) \\ &= i \int_t^T e^{i\sqrt{\mu}(t-s)} (-iw'(s) + \sqrt{\mu} w(s))\, ds. \end{aligned} \tag{5}$$

3.2 A convenient REALization

We are only interested in testing with real valued functions v. However, for any real valued function $w \in H^1_{0,}(0,T)$, we see from (5) that $\mathcal{T}_\mu w$ is complex valued. To circumvent the use of complex functions, we now consider only the real part of the transformation operator and define for $w \in H^1_{0,}(0,T)$

$$\mathsf{T}_\mu w(t) := \Re(\mathcal{T}_\mu w(t)) = \int_t^T \left[\cos(\sqrt{\mu}(t-s)) w'(s) - \sqrt{\mu} \sin(\sqrt{\mu}(t-s)) w(s)\right] ds.$$

Lemma 1 *The operator $\mathsf{T}_\mu : H^1_{0,}(0,T) \to H^1_{,0}(0,T)$ is well-defined, bounded and satisfies for all $u \in H^1_{0,}(0,T)$ and all $q \in L^2(0,T)$*

$$\int_0^T \left[-(\mathsf{T}_\mu u)'(t) + \mu \int_t^T (\mathsf{T}_\mu u)(s)\, ds\right] q(t)\, dt = \int_0^T \left[u'(t) + \mu \int_t^T u(s)\, ds\right] q(t)\, dt.$$

Proof. By construction $(\mathsf{T}_\mu u)(T) = 0$ and we compute

$$(\mathsf{T}_\mu u)'(t) = -u'(t) - \sqrt{\mu} \int_t^T \left[\sin(\sqrt{\mu}(t-s)) u'(s) + \sqrt{\mu} \cos(\sqrt{\mu}(t-s)) u(s)\right] ds.$$

For fixed μ, all the terms on the right-hand side are bounded in $L^2(0,T)$ if $u \in H^1_{0,}(0,T)$, and thus T_μ maps into $H^1_{,0}(0,T)$. Using a triangle and Cauchy–Schwarz inequalities, we also get the bound

$$\left\|\mathsf{T}_\mu u\right\|_{H^1_{,0}(0,T)} \leq \left(1 + T\sqrt{\frac{\mu}{2}}\right) \|u\|_{H^1_\mu}, \quad u \in H^1_{0,}(0,T). \tag{6}$$

Furthermore, using that $(\mathsf{T}_\mu u)'(T) = -u'(T)$, we compute that

$$(\mathsf{T}_\mu u)'' + \mu\, \mathsf{T}_\mu u = -u'' + \mu u \ \text{ in } [H^1_{0,}(0,T)]'.$$

Hence, for all $w \in H^1_{0,}(0,T)$ we have

$$-\langle (\mathsf{T}_\mu u)', w' \rangle + \mu \langle \mathsf{T}_\mu u, w \rangle = \langle u', w' \rangle + \mu \langle u, w \rangle,$$

where we applied the integration by parts formula (3). Next, we use that each $w \in H^1_{0,}(0,T)$ admits the representation $w(t) = \int_0^t q(s)\, ds$ for some $q \in L^2(0,T)$. Thus, $w' = q$ and using the following integration by parts formula

$$\int_0^T p(t) \int_0^t q(s)\, ds\, dt = \int_0^T \int_t^T p(s)\, ds\, q(t)\, dt, \quad \forall p, q \in L^2(0,T), \tag{7}$$

concludes the proof. □

Remark 1 We like to point out that T_μ is invertible. Moreover, note that for $\mu = 0$, $\mathsf{T}_0\, w(t) = w(T) - w(t) = \overline{\mathcal{H}}_T w(t)$, which was introduced in [8, Lemma 4.5] as a purely temporal transformation for the stabilization of the wave equation. T_μ can thus be seen as an extension of $\overline{\mathcal{H}}_T$ to space-time.

4 Always Look on the Right Side of Life

Using the real transformation T_μ, we are now considering the variational formulation: Given $f \in [H^1_{,0}(0,T)]'$, find $u \in H^1_{0,}(0,T)$ such that

$$b_\mu(u, \mathsf{T}_\mu w) = \langle f, \mathsf{T}_\mu w \rangle, \quad \forall w \in H^1_{0,}(0,T). \tag{8}$$

We first prove that its application on the second argument of b_μ does not introduce complicated expressions or additional terms.

Lemma 2 *For all $u, w \in H^1_{0,}(0,T)$ it holds that*

$$b_\mu(u, \mathsf{T}_\mu w) = \langle u', w' \rangle + \mu \langle u, w \rangle.$$

In particular, b_μ is bounded and T_μ-coercive, i.e.,

$$b_\mu(u, \mathsf{T}_\mu w) \leq \|u\|_{H^1_\mu} \|w\|_{H^1_\mu}, \quad \textit{and} \quad b_\mu(u, \mathsf{T}_\mu u) = \|u\|^2_{H^1_\mu}.$$

Proof. As $u \in H^1_{0,}(0,T)$ there exists $q \in L^2(0,T)$ such that $u(t) = \int_0^t q(s)\,ds$. Using (7) and Lemma 1, we then compute for all $w \in H^1_{0,}(0,T)$

$$\begin{aligned} b_\mu(u, \mathsf{T}_\mu w) &= \int_0^T \left[-u'(t)(\mathsf{T}_\mu w)'(t) + \mu u(t)(\mathsf{T}_\mu w)(t)\right]\,dt \\ &= \int_0^T q(t)\left(-(\mathsf{T}_\mu w)'(t) + \mu \int_t^T (\mathsf{T}_\mu w)(s)\,ds\right)\,dt \\ &= \int_0^T q(t)\left(w'(t) + \mu \int_t^T w(s)\,ds\right)\,dt \\ &= \int_0^T u'(t)w'(t) + \mu u(t)w(t)\,dt. \end{aligned}$$

□

The main result is now the following.

Theorem 2 *The variational formulation* (8) *admits a unique solution* $u \in H^1_{0,}(0,T)$ *for all* $f \in [H^1_{,0}(0,T)]'$. *In particular, for all*

$$f \in \mathcal{F} := \{f \in [H^1_{,0}(0,T)]' : \langle f, \mathsf{T}_\mu w\rangle \le C\|w\|_{H^1_\mu},\ \forall w \in H^1_{0,}(0,T)\}.$$

with some $C > 0$ *independent of* μ, *the solution* u *satisfies the* μ*-independent bound*

$$\|u\|_{H^1_\mu} \le C. \tag{9}$$

Proof. By Lemma 2 the bilinear form b_μ is T_μ-coercive and bounded. Moreover, for fixed $\mu > 0$, using (6) we get

$$\langle f, \mathsf{T}_\mu w\rangle \le \|f\|_{[H^1_{,0}(0,T)]'}\left\|\mathsf{T}_\mu w\right\|_{H^1_{,0}(0,T)} \le \left(1 + T\sqrt{\frac{\mu}{2}}\right)\|f\|_{[H^1_{,0}(0,T)]'}\|w\|_{H^1_\mu}.$$

Hence, by the Lemma of Lax-Milgram, there exists a unique solution $u \in H^1_{0,}(0,T)$ for all $f \in [H^1_{,0}(0,T)]'$. The bound (9) for $f \in \mathcal{F}$ now follows from

$$\|u\|^2_{H^1_\mu} = b_\mu(u, \mathsf{T}_\mu u) = \langle f, \mathsf{T}_\mu u\rangle \le C\|u\|_{H^1_\mu}.$$

□

Remark 2 For $w \in H^1_{0,}(0,T)$, using (7) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \langle f, \mathsf{T}_\mu w\rangle &= \int_0^T \left[w'(t)\,\mathsf{C}(t,f) + \sqrt{\mu}w(t)\,\mathsf{S}(t,f)\right]\,dt \\ &\le \|w\|_{H^1_\mu}\left(\int_0^T \left[\mathsf{C}(t,f)^2 + \mathsf{S}(t,f)^2\right]\,dt\right)^{1/2} \end{aligned}$$

where $\mathsf{C}(t,f) = \int_0^t \cos(\sqrt{\mu}(t-s))f(s)\,ds$ and $\mathsf{S}(t,f) = \int_0^t \sin(\sqrt{\mu}(t-s))f(s)\,ds$. For $f \in L^2(0,T)$ we can now proceed as in [8, Lemma 4.5], to compute

$$\int_0^T \left[\mathsf{C}(t,f)^2 + \mathsf{S}(t,f)^2\right]\, dt \le \frac{T^2}{2}\|f\|^2_{L^2(0,T)},$$

showing that $L^2(0,T) \subset \mathcal{F}$ and (9) holds with $C = \frac{T}{\sqrt{2}}\|f\|_{L^2(0,T)}$.

5 Discretization

We consider the conforming trial space $X_h := S_h^1(0,T) \cap H^1_{0,}(0,T)$ of piecewise linear finite elements defined on a uniform decomposition of the interval $(0,T)$ into $N \in \mathbb{N}$ elements of mesh size $h = T/N$. The discrete variational formulation reads: Given $f \in [H^1_{,0}(0,T)]'$, find $u_h \in X_h$ such that

$$b_\mu(u_h, \mathsf{T}_\mu\, w_h) = \langle u_h', w_h'\rangle + \mu\langle u_h, w_h\rangle = \langle f, \mathsf{T}_\mu\, w_h\rangle,\ \forall w_h \in X_h. \tag{10}$$

Due to the T_μ-coercivity, (10) admits a unique solution $u_h \in X_h$. By Lemma 2 and the linearity of T_μ we have Galerkin orthogonality and immediately derive Cea's Lemma and thus the best-approximation error estimate for $u \in H^s(0,T)$

$$\|u - u_h\|_{H^1_\mu} \le \inf_{w_h \in X_h} \|u - w_h\|_{H^1_\mu} \le c(h^{s-1} + \sqrt{\mu}h^s)|u|_{H^s(0,T)},\ s \in [1,2].$$

As an illustrative example, we consider $T = 1$, and the function

$$u(t) = t^2(T-t)^{3/4} \in H^s(0,T),\ s < \frac{5}{4}$$

solving (1) for $f \in H^{-\sigma}(0,T)$, $\sigma > \frac{3}{4}$. The solutions for $\mu \in \{1, 10^5\}$ are depicted in Figure 1, and the convergence rates are listed in Table 1. We see, that the method is stable w.r.t. μ and gives optimal orders of convergence in H^1 right from the start. Using analytic integration, the load vector can be computed with effort $O(N^2 \cdot p)$, where p is the number of quadrature points, depending on the smoothness of f.

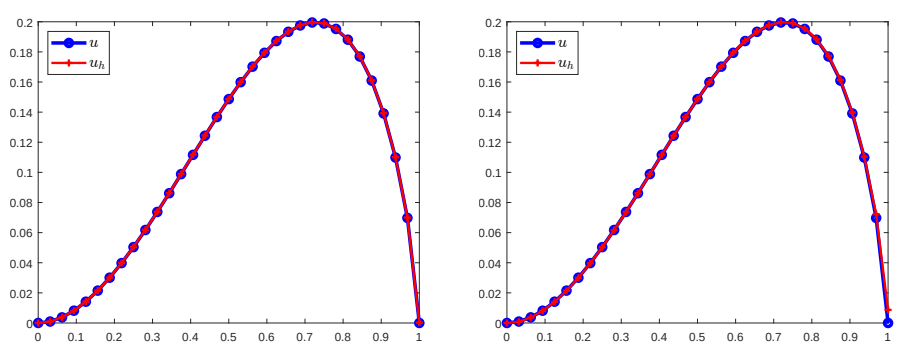

Fig. 1 Exact solution u and reconstruction u_h for $\mu = 1$ (left) and $\mu = 10^5$ (right) on $N = 32$ elements.

		$\mu = 1$				$\mu = 1000$		$\mu = 10^5$	
N	h	$\|u - u_h\|_{L^2}$	eoc	$\|u - u_h\|_{H^1}$	eoc	$\|u - u_h\|_{H^1}$	eoc	$\|u - u_h\|_{H^1}$	eoc
4	0.250	2.13e-02	0.00	3.19e-01	0.00	3.29e-01	0.00	3.30e-01	0.00
8	0.125	7.70e-03	1.47	2.23e-01	0.52	2.26e-01	0.54	2.29e-01	0.53
16	0.063	2.89e-03	1.41	1.62e-01	0.46	1.63e-01	0.47	1.67e-01	0.46
32	0.031	1.13e-03	1.35	1.23e-01	0.39	1.24e-01	0.40	1.27e-01	0.39
64	0.016	4.57e-04	1.31	9.80e-02	0.33	9.80e-02	0.34	1.00e-01	0.34
128	0.008	1.88e-04	1.28	7.99e-02	0.30	7.98e-02	0.30	8.08e-02	0.31
256	0.004	7.81e-05	1.27	6.60e-02	0.27	6.60e-02	0.27	6.62e-02	0.29
512	0.002	3.26e-05	1.26	5.51e-02	0.26	5.51e-02	0.26	5.51e-02	0.28

Table 1 Errors and orders of convergence for different $\mu \in \{1, 1000, 10^5\}$.

6 Conclusions

We proposed a novel transformation operator for an ODE that is related to a space-time FEM formulation of the wave equation, resulting in a Galerkin-Bubnov formulation that is unconditionally stable and coercive. We can theoretically prove stability and best approximation error estimates independent of μ. This opens the door for a space-time transformation that leads to a coercive formulation. Related results will be published elsewhere.

Acknowledgements The research leading to this publication received funding from the Dutch Research Council (NWO) under the NWO-Talent Programme Veni with the project number VI.Veni.212.253.

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