

Uzawa Algorithm for Nonhomogeneous Navier-Stokes Type Problems

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1 Introduction

The literature on the Navier-Stokes problem is abundant. We only cite here a few books almost entirely devoted to this problem, [1], [2], [3] and [4]. Other books contain chapters dealing with various aspects of the problem. In [5], for example, the time-dependent problem is approached from the point of view of nonlinear semigroups. However, to our knowledge, a systematic analysis of the Uzawa and Arrow–Hurwicz methods or their variants is lacking in the literature, even though they are the main numerical methods used to solve the Stokes and Navier-Stokes problems. We tried to overcome this gap in [6] and [7] in the case of the saddle point problems, therefore for Stokes problems as well.

In this paper we adapt to the nonhomogeneous case the algorithm that has been introduced in [3], page 218, for the resolution of the homogeneous Navier-Stokes problems. In the next section, we write the nonhomogeneous Navier-Stokes problem in a form suitable for the analysis of the Uzawa algorithm and prove or recall results that will be helpful in proving the convergence of algorithm. In Section 3, the Uzawa algorithm is defined for a more general than the usual form of the Navier-Stokes problem, the incompressibility condition being replaced by a variational inequality. The convergence theorem has two parts. We first analyze the convergence of the algorithm without considering that the inf-sup condition is satisfied and we obtain that the velocity strongly converges and the pressure is weakly convergent. Next, we introduce the inf-sup condition and prove that both the velocity and the pressure are strongly convergent and write explicit formulas for their convergence rates.

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2 Framework for the nonhomogeneous Navier-Stokes problem

For a given bounded domain Ω in \mathbf{R}^d , $d \leq 4$, with a Lipschitz continuous boundary Γ , we consider the following nonhomogeneous Navier-Stokes problem: find the velocity \mathbf{u} and the pressure p satisfying

$$\begin{aligned} -\nu\Delta\mathbf{u} + \sum_{j=1}^n \mathbf{u}_j \frac{\partial\mathbf{u}}{\partial x_j} + \mathbf{grad} p &= \mathbf{f} \text{ in } \mathbf{H}^{-1}(\Omega) \\ \operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma \end{aligned} \quad (1)$$

where ν is the viscosity of the fluid, $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ such that $\int_{\Gamma} \mathbf{g} \cdot \mathbf{n} = 0$, where \mathbf{n} is the unit exterior normal along Γ . From Theorem 3.5 of [2], page 32, we get that \mathbf{g} can be extended on entire Ω such that

$$\mathbf{g} \in \mathbf{H}^1(\Omega) \text{ and } \operatorname{div} \mathbf{g} = 0 \text{ in } \Omega \quad (2)$$

We write $\hat{\mathbf{u}} = \mathbf{u} - \mathbf{g}$, and then problem (1) reads

$$\begin{aligned} -\nu\Delta\hat{\mathbf{u}} + \sum_{j=1}^d \hat{\mathbf{u}}_j \frac{\partial\hat{\mathbf{u}}}{\partial x_j} + \sum_{j=1}^d \hat{\mathbf{u}}_j \frac{\partial\mathbf{g}}{\partial x_j} + \sum_{j=1}^d \mathbf{g}_j \frac{\partial\hat{\mathbf{u}}}{\partial x_j} + \mathbf{grad} p &= \hat{\mathbf{f}} \text{ in } \mathbf{H}^{-1}(\Omega) \\ \operatorname{div} \hat{\mathbf{u}} &= 0 \text{ in } \Omega, \quad \hat{\mathbf{u}} = 0 \text{ on } \Gamma \end{aligned} \quad (3)$$

where $\hat{\mathbf{f}} = \mathbf{f} + \nu\Delta\mathbf{g} - \sum_{j=1}^d \mathbf{g}_j \frac{\partial\mathbf{g}}{\partial x_j}$.

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$, we denote

$$a(\mathbf{u}, \mathbf{v}) = \nu \sum_{i=1}^d \int_{\Omega} \mathbf{grad} u_i \cdot \mathbf{grad} v_i, \quad b(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \sum_{i=1}^d \sum_{j=1}^d \int_{\Omega} \mathbf{w}_j \frac{\partial u_i}{\partial x_j} v_i \quad (4)$$

and define the variational formulation of problem (3) as: find $\hat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} a(\hat{\mathbf{u}}, \mathbf{v}) + b(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) + b(\hat{\mathbf{u}}, \mathbf{g}, \mathbf{v}) + b(\hat{\mathbf{g}}, \hat{\mathbf{u}}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} \\ = \langle \hat{\mathbf{f}}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ (q, \operatorname{div}(\hat{\mathbf{u}}))_{L^2(\Omega)} = 0 \text{ for any } q \in L_0^2(\Omega) \end{aligned} \quad (5)$$

We know (see Theorem 3.7 of [2], page 35) that, in case of our choice of spaces, there is a constant $\beta > 0$ for which the inf-sup condition

$$\inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(q, \operatorname{div} \mathbf{v})_{L^2(\Omega)}}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta \quad (6)$$

is satisfied. Also, from Lemma 2.1 of [2], page 114, it follows that there exists $\mathcal{N} > 0$ such that

$$b(\mathbf{w}, \mathbf{u}, \mathbf{v}) \leq \mathcal{N} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \text{ for any } \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (7)$$

Now, let us define $d : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$d(\mathbf{w}, \mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{g}, \mathbf{v}) + b(\mathbf{g}, \mathbf{u}, \mathbf{v}) \quad (8)$$

and let $D : \mathbf{H}_0^1(\Omega) \rightarrow \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{H}^{-1}(\Omega))$ be defined as

$$\langle (D\mathbf{w})(\mathbf{u}), \cdot \rangle = a(\mathbf{u}, \cdot) + b(\mathbf{w}, \mathbf{u}, \cdot) + b(\mathbf{u}, \mathbf{g}, \cdot) + b(\mathbf{g}, \mathbf{u}, \cdot) \quad (9)$$

Using these notations, we have

Lemma 1 *Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that $\operatorname{div}(\mathbf{w}) = 0$ and let us assume that*

$$|b(\mathbf{v}, \mathbf{g}, \mathbf{v})| \leq \frac{\nu}{2} |\mathbf{v}|_{\mathbf{H}^1(\Omega)}^2 \text{ for any } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad (10)$$

where $\mathbf{g} \in \mathbf{H}^1(\Omega)$ satisfies (2) and gives the boundary value of the nonhomogeneous Navier-Stokes problem (1). Then the bilinear form $\langle (D\mathbf{w})\cdot, \cdot \rangle$ is uniformly elliptic on $\mathbf{H}_0^1(\Omega)$,

$$\langle (D\mathbf{w})(\mathbf{v}), \mathbf{v} \rangle \geq \frac{\nu}{2} |\mathbf{v}|_{\mathbf{H}^1(\Omega)}^2 \text{ for any } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (11)$$

and is invertible. Besides that, for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}_0^1(\Omega)$, we have

$$\|D\mathbf{w}_1 - D\mathbf{w}_2\|_{\mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{H}^{-1}(\Omega))} \leq \mathcal{N} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{H}^1(\Omega)} \quad (12)$$

Proof. Equation (11) follows from (10), from the ellipticity of the bilinear form a , $a(\mathbf{v}, \mathbf{v}) \geq \nu |\mathbf{v}|^2$ and from $b(\mathbf{w}, \mathbf{v}, \mathbf{v}) = b(\mathbf{g}, \mathbf{v}, \mathbf{v}) = 0$, since $\operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{g} = 0$. From the ellipticity property (11) of $D\mathbf{w}$ and the Lax-Milgram lemma, we get that $D\mathbf{w}$ is invertible. From the definition of $D\mathbf{w}$ and (7), for any $\mathbf{v}, \mathbf{u} \in \mathbf{H}_0^1(\Omega)$, we have

$$\begin{aligned} \langle ((D\mathbf{w}_1) - (D\mathbf{w}_2))(\mathbf{u}), \mathbf{v} \rangle &= b(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{u}, \mathbf{v}) \\ &\leq \mathcal{N} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \end{aligned} \quad (13)$$

for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}_0^1(\Omega)$, i.e., (12) holds true. \square

Condition (10) has been introduced in [3], see Theorem 1.6, page 178. For $d \leq 4$ we can write (7) as

$$\begin{aligned} b(\mathbf{w}, \mathbf{g}, \mathbf{v}) &\leq \mathcal{N} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{g}\|_{\mathbf{L}^4(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \\ &\text{for any } \mathbf{w}, \mathbf{v} \in \mathbf{H}^1(\Omega) \text{ and } \mathbf{g} \in \mathbf{L}^4(\Omega) \end{aligned} \quad (14)$$

Therefore, condition (10) supposes that the norm of \mathbf{g} in $\mathbf{L}^4(\Omega)$ can be chosen sufficiently small. Using the above notations, Navier-Stokes problem (5) can be written as: find $\hat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\langle (D\hat{\mathbf{u}})(\hat{\mathbf{u}}), \mathbf{v} \rangle + \langle \mathbf{grad} p, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad (15)$$

$$(\operatorname{div} \hat{\mathbf{u}}, q)_{L^2(\Omega)} = 0 \text{ for any } q \in L_0^2(\Omega) \quad (16)$$

where $L_0^2(\Omega) \ni p \mapsto \mathbf{grad} p \in \mathbf{H}^{-1}(\Omega)$ is taken in the sense of distributions. Now, to prove the existence and uniqueness of the solution of this problem, using condition (10) and Lemma 1, we can follow a similar way as in the proof of Theorem 2.5 in [2], page 118. We get that if $\frac{4N}{\nu^2} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} < 1$, is satisfied, then problem (15)-(16) has a unique solution $(\hat{\mathbf{u}}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$. Also, in view of (11) and (15), we get

$$\|\hat{\mathbf{u}}\|_{\mathbf{H}_1(\Omega)} \leq \frac{2}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} \quad (17)$$

and using (9) and (17), we have,

$$\|D\hat{\mathbf{u}}\|_{\mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{H}^{-1}(\Omega))} \leq \nu + N \left(\frac{2}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} + 2\|\mathbf{g}\|_{\mathbf{H}(\Omega)} \right) \quad (18)$$

3 Uzawa algorithm for a more general problem

Since the sequence of \mathbf{u}^n obtained from the Uzawa algorithm does not satisfy anymore condition $b(\hat{\mathbf{u}}^n, \mathbf{v}, \mathbf{v}) = 0$ for $\mathbf{v} \in H_0^1(\Omega)$, it is proposed in [3], page 205, to replace $b(\mathbf{w}, \mathbf{u}, \mathbf{v})$ by

$$\hat{b}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_{\Omega} \mathbf{w}_j \frac{\partial \mathbf{u}_i}{\partial x_j} \mathbf{v}_i - \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_{\Omega} \mathbf{w}_j \frac{\partial \mathbf{v}_i}{\partial x_j} \mathbf{u}_i$$

for any $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$

It is easy to see that

$$\begin{aligned} \hat{b}(\mathbf{w}, \mathbf{u}, \mathbf{v}) &= b(\mathbf{w}, \mathbf{u}, \mathbf{v}) \text{ for any } \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \text{ and } \operatorname{div} \mathbf{w} = 0 \\ \hat{b}(\mathbf{w}, \mathbf{v}, \mathbf{v}) &= 0 \text{ for any } \mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \end{aligned}$$

and therefore, the results of the previous section hold true if we consider $\hat{b}(\mathbf{w}, \mathbf{u}, \mathbf{v})$ in the place of $b(\mathbf{w}, \mathbf{u}, \mathbf{v})$ and consider the operator $\hat{D} : \mathbf{H}_0^1(\Omega) \rightarrow \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{H}^{-1}(\Omega))$ defined by

$$\langle (\hat{D}\mathbf{w})(\mathbf{u}), \cdot \rangle = a(\mathbf{u}, \cdot) + \hat{b}(\mathbf{w}, \mathbf{u}, \cdot) + b(\mathbf{u}, \mathbf{g}, \cdot) + b(\mathbf{g}, \mathbf{u}, \cdot)$$

in the place of the operator D defined in (9).

Since the operator \hat{D} does not need the incompressibility condition to satisfy the properties of D in the previous section, we consider the Uzawa algorithm associated with a more general problem than the Navier-Stokes problem defined in (15)-(16),

$$\langle \hat{D}\hat{\mathbf{u}}(\hat{\mathbf{u}}), \mathbf{v} \rangle + \langle \mathbf{grad} p, \mathbf{v} \rangle = \langle \hat{\mathbf{f}}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad (19)$$

$$\langle \mathbf{grad} (q - p), \hat{\mathbf{u}} \rangle \leq 0 \text{ for any } q \in K \quad (20)$$

where K is a closed convex set of $L_0^2(\Omega)$. We assume that this problem has at least a solution $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times K$, the component \mathbf{u} being unique.

Let us consider a linear symmetric operator $S : L_0^2(\Omega) \rightarrow L_0^2(\Omega)'$ which is $L_0^2(\Omega)$ -elliptic and continuous, i.e., there exist $\sigma, \eta > 0$ such that

$$\eta \|q\|_{L_0^2(\Omega)}^2 \leq \langle S(q), q \rangle \text{ and } \|S(q)\|_{L_0^2(\Omega)'} \leq \sigma \|q\|_{L_0^2(\Omega)} \text{ for any } q \in L_0^2(\Omega) \quad (21)$$

where $L_0^2(\Omega)'$ is the dual space of $L_0^2(\Omega)$. The inner product and the norm associated with S are denoted by $(p, q)_S = \langle Sp, q \rangle$ and $\|p\|_S = (p, p)_S^{1/2}$ for any $p, q \in L_0^2(\Omega)$, respectively.

We write the Uzawa algorithm associated with problem (19)-(20) as: we start with arbitrary $\hat{\mathbf{u}}^0 \in \mathbf{H}_0^1(\Omega)$ and $p^0 \in K$ and at iteration $n + 1$, $n \geq 0$, having $\hat{\mathbf{u}}^n \in \mathbf{H}_0^1(\Omega)$ and $p^n \in K$ we find $\hat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and $p^{n+1} \in K$ by solving

$$\hat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega) : \langle (\hat{D}\hat{\mathbf{u}}^{n+1})(\hat{\mathbf{u}}^{n+1}), \mathbf{v} \rangle + \langle \mathbf{grad} p^n, \mathbf{v} \rangle = \langle \hat{\mathbf{f}}, \mathbf{v} \rangle \quad (22)$$

for any $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$

$$p^{n+1} \in K : \langle S(p^{n+1}), q - p^{n+1} \rangle \geq \langle S(p^n), q - p^{n+1} \rangle + \rho \langle \mathbf{grad} (q - p^{n+1}), \hat{\mathbf{u}}^{n+1} \rangle \text{ for any } q \in K \quad (23)$$

where ρ is a positive constant which will be chosen later on.

Theorem 1 Let $(\hat{\mathbf{u}}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be a solution of problem (19)-(20) and $(\hat{\mathbf{u}}^n, p^n) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, $n \geq 1$, are obtained from algorithm (22)-(23). We assume that

$$\rho < \eta \nu \left(1 - \frac{4N}{\nu^2} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} \right) \quad (24)$$

holds. Then,

a) if we assume that the inf-sup condition (6) does not hold, then $\hat{\mathbf{u}}^n \rightarrow \hat{\mathbf{u}}$ as $n \rightarrow \infty$, strongly in $\mathbf{H}_0^1(\Omega)$, and any subsequence of $(p^n)_n$ contains a subsequence $(p^{n_k})_{n_k}$ and there exists a $p \in L_0^2(\Omega)$ such that $p^{n_k} \rightharpoonup p$ as $n_k \rightarrow \infty$, weakly in $L_0^2(\Omega)$. If problem (19)-(20) has a unique solution, then $(\hat{\mathbf{u}}, p)$,

b) if we assume that the inf-sup condition (6) is satisfied, then there exists $\hat{N} \geq 1$ such that we have the convergence estimates

$$\|p - p^n\|_S^2 \leq C_{\text{rate}}^{n-\hat{N}} \|p - p^{\hat{N}}\|_S^2 \quad (25)$$

$$\|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \leq C_{\text{rate}}^{n-\hat{N}} \frac{\|p - p^{\hat{N}}\|_S^2}{\rho \nu \left(1 - \frac{4N}{\nu^2} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{\nu \eta} \right)} \quad (26)$$

for any $b \geq \hat{N}$, where

$$C_{\text{rate}} = 1 - \rho \frac{\nu}{C_f^2} \left(1 - \frac{4N}{\nu^2} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{\nu \eta} \right) \text{ and} \quad (27)$$

$$C_f = \frac{\sigma^{1/2}}{\beta} \left(N + \frac{2N}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu + 2N \|\mathbf{g}\|_{\mathbf{H}^1(\Omega)} \right) \quad (28)$$

Proof. a) From (19) and (22), we get

$$\langle (\hat{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle - \langle (\hat{D}\hat{\mathbf{u}}^{n+1})(\hat{\mathbf{u}}^{n+1}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle + \langle \mathbf{grad} (p - p^n), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle = 0$$

or

$$\langle (\hat{D}\hat{\mathbf{u}} - \hat{D}\hat{\mathbf{u}}^{n+1})(\hat{\mathbf{u}}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle + \langle (\hat{D}\hat{\mathbf{u}}^{n+1})(\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle + \langle \mathbf{grad} (p - p^n), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle = 0$$

and using (11) and (12), we have

$\frac{\nu}{2} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \leq \mathcal{N} \|\hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 - \langle \mathbf{grad}(p - p^n), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle$
and, in view of (17), we have

$$\frac{\nu}{2} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \leq \frac{2\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 - \langle \mathbf{grad}(p - p^n), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle \quad (29)$$

From (20) and (23), we have

$$\langle S(p^{n+1}) - S(p), p - p^{n+1} \rangle \geq \langle S(p^n) - S(p), p - p^{n+1} \rangle + \rho \langle \mathbf{grad}(p - p^{n+1}), \hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}} \rangle$$

or

$$\|p - p^{n+1}\|_S^2 \leq (p - p^n, p - p^{n+1})_S + \rho \langle \mathbf{grad}(p - p^{n+1}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle \quad (30)$$

i.e., since

$$2\|p - p^{n+1}\|_S^2 - 2(p - p^n, p - p^{n+1})_S = \|p - p^{n+1}\|_S^2 + \|p^n - p^{n+1}\|_S^2 - \|p - p^n\|_S^2,$$

we have

$$\|p - p^{n+1}\|_S^2 + \|p^n - p^{n+1}\|_S^2 - \|p - p^n\|_S^2 \leq 2\rho \langle \mathbf{grad}(p - p^{n+1}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle \quad (31)$$

In view of (29) and (31), we get

$$\begin{aligned} & \frac{\nu}{2} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + \frac{1}{2\rho} \|p - p^{n+1}\|_S^2 + \frac{1}{2\rho} \|p^n - p^{n+1}\|_S^2 - \frac{1}{2\rho} \|p - p^n\|_S^2 \\ & \leq \frac{2\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + \langle \mathbf{grad}(p^n - p^{n+1}), \hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \rangle \end{aligned} \quad (32)$$

and therefore, using the fact that $\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and consequently, $\|\operatorname{div}(\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1})\|_{L^2(\Omega)} \leq \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}$ (see Lemma 2.1 (Div-Grad Relation) of [8]), we have

$$\begin{aligned} & \left(\frac{\nu}{2} - \frac{2\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} \right) \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \\ & + \frac{1}{2\rho} \|p - p^{n+1}\|_S^2 + \frac{1}{2\rho} \|p^n - p^{n+1}\|_S^2 - \frac{1}{2\rho} \|p - p^n\|_S^2 \\ & \leq \frac{\delta_1}{2} \|p^n - p^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2\delta_1} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \end{aligned} \quad (33)$$

for any $\delta_1 > 0$. Finally, using (21), we have

$$\begin{aligned} & \left(\frac{\nu}{2} - 2\frac{\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{1}{2\delta_1} \right) \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + \frac{1}{2\rho} \|p - p^{n+1}\|_S^2 \\ & + \left(\frac{1}{2\rho} - \frac{\delta_1}{2\eta} \right) \|p^n - p^{n+1}\|_S^2 \leq \frac{1}{2\rho} \|p - p^n\|_S^2 \end{aligned} \quad (34)$$

For $\delta_1 = \frac{\eta}{\rho}$, we get

$$\begin{aligned} & \left(\frac{\nu}{2} - 2\frac{\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{2\eta} \right) \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \\ & + \frac{1}{2\rho} \|p - p^{n+1}\|_S^2 \leq \frac{1}{2\rho} \|p - p^n\|_S^2 \end{aligned} \quad (35)$$

For any $\tilde{N} \geq 1$, summing the above equation for $n = 0, \dots, \tilde{N} - 1$, we get

$$\left(\frac{\nu}{2} - 2\frac{\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{2\eta} \right) \sum_{n=1}^{\tilde{N}} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^n\|_{\mathbf{H}^1(\Omega)}^2 + \frac{1}{2\rho} \|p - p^{\tilde{N}}\|_S^2 \leq \frac{1}{2\rho} \|p - p^0\|_S^2$$

and therefore, in view (24), we get that series $\sum_{n=1}^{\infty} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}^n\|_{\mathbf{H}^1(\Omega)}^2$ is convergent and the sequence $(\|p - p^n\|_S^2)_n$ is bounded. Therefore, $\hat{\mathbf{u}}^n \rightarrow \hat{\mathbf{u}}$ in $\mathbf{H}^1(\Omega)$ and the sequence of p_n , $n \geq 1$ is bounded in $L^2(\Omega)$, i.e. the conclusion of item a) in the statement of the theorem holds.

b) Now, we will use the inf-sup condition (6). From (19) and (22), we get

$$\langle (\hat{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}), \mathbf{v} \rangle - \langle (\hat{D}\hat{\mathbf{u}}^{n+1})(\hat{\mathbf{u}}^{n+1}), \mathbf{v} \rangle + \langle \mathbf{grad}(p - p^n), \mathbf{v} \rangle = 0$$

and we have

$$\begin{aligned} & -\langle (\hat{D}\hat{\mathbf{u}}^{n+1} - \hat{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}), \mathbf{v} \rangle - \langle (\hat{D}\hat{\mathbf{u}}^{n+1} - \hat{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}), \mathbf{v} \rangle \\ & -\langle (\hat{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}), \mathbf{v} \rangle = -\langle \mathbf{grad}(p - p^n), \mathbf{v} \rangle \end{aligned}$$

In view of (6), (12), (17) and (18), we get

$$\begin{aligned} \beta \|p - p^n\|_{L^2(\Omega)} & \leq \mathcal{N} \|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}^2 \\ & + \left(\frac{4\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu + 2\mathcal{N} \|\mathbf{g}\|_{\mathbf{H}^1(\Omega)} \right) \|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} \end{aligned}$$

In view of item a), we get that if (24) holds, there exists an $\hat{N} > 0$ such that $\|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} \leq 1$ for any $n \geq \hat{N}$. It follows from the above equation that

$$\beta \|p - p^n\|_{L^2(\Omega)} \leq \left(\mathcal{N} + \frac{4\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu + 2\mathcal{N} \|\mathbf{g}\|_{\mathbf{H}^1(\Omega)} \right) \|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}$$

and using (21), we have

$$\|p - p^n\|_S^2 \leq \frac{\sigma}{\beta^2} \left(\mathcal{N} + \frac{4\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu + 2\mathcal{N} \|\mathbf{g}\|_{\mathbf{H}^1(\Omega)} \right)^2 \|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}^2$$

for any $n \geq \hat{N}$. Using C_f defined in (28), we have

$$\|p - p^n\|_S^2 \leq C_f^2 \|\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}^2 \quad (36)$$

for any $n \geq \hat{N}$. Combining (35) with (36), we have

$$\|p - p^{n+1}\|_S^2 \leq \left[1 - \frac{2\rho}{C_f^2} \left(\frac{\nu}{2} - 2\frac{\mathcal{N}}{\nu} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{2\eta} \right) \right] \|p - p^n\|_S^2 \quad (37)$$

for any $n \geq \hat{N}$. Therefore, we got (25). Now, from (35), we have

$$\|\hat{\mathbf{u}} - \hat{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 \leq \frac{1}{\rho\nu \left(1 - \frac{4\mathcal{N}}{\nu^2} \|\hat{f}\|_{\mathbf{H}^{-1}(\Omega)} - \frac{\rho}{\nu\eta} \right)} \|p - p^n\|_S^2 \quad (38)$$

and using (25), we get (26). □

Remark 1 1) Denoting $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{g}$ and $\mathbf{u}^n = \hat{\mathbf{u}}^n + \mathbf{g}$, problem (19)-(20) will become a nonhomogeneous Navier-Stokes type problem where the incompressibility condition has been replaced by an inequality, with the solution (\mathbf{u}, p) , and algorithm (22)-(23) will become its associated Uzawa algorithm with, the solution (\mathbf{u}^n, p^n) , $n \geq 0$. In view of this remark, Theorem 1 can be easily rewritten and applied to the convergence of the Uzawa algorithm for this nonhomogeneous Navier-Stokes type problem. Evidently, this remark also holds for the usual nonhomogeneous Navier-Stokes problem

and its associated Uzawa algorithm, since, in this case, that inequalities (20) and (23) are written for $K = L_0^2(\Omega)$.

2) We also note that the above results remain valid for problems formulated for pairs of finite element spaces that have the properties of $\mathbf{H}_0^1(\Omega)$ and $L_0^2(\Omega)$ requested in the manuscript.

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