Multiscale Spectral Generalized Finite Element Methods for Discontinuous Galerkin Schemes

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1 Motivation and Model Setup

This article presents a multiscale spectral generalized finite element method (MS-GFEM) tailored to discontinuous Galerkin (DG) discretizations of partial differential equations (PDEs). Our work is motivated by two problem classes: heterogeneous Stokes flows, where DG discretizations improve mass conservation, and convection-dominated diffusion, where DG fluxes enhance stability. Previous work established nearly exponential error decay for MS-GFEM in the Hilbert-space setting [3] and for continuous Galerkin methods [4]. As a first step toward extending that analysis to DG discretizations, we prove nearly exponential decay of the MS-GFEM approximation error for second-order elliptic PDEs with highly heterogeneous diffusion discretized by a weighted symmetric interior-penalty discontinuous Galerkin method.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a polygonal Lipschitz domain. We consider the following second-order elliptic PDE with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\operatorname{div}(\nu \nabla u) = f & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and $\nu \in L^\infty(\Omega)$. We assume that $\nu_{\min} \leq \nu(\boldsymbol{x}) \leq \nu_{\max}$ holds for almost all $\boldsymbol{x} \in \Omega$ for some constants $\nu_{\max}, \nu_{\min} > 0$. The weak form of (1) is based on the Sobolev space $H_0^1(\Omega)$ of H^1-functions vanishing on the boundary $\partial\Omega$. Given $f \in L^2(\Omega)$, the weak form of (1) seeks a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nu \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x =: \mathcal{F}(v) \qquad \forall\, v \in H_0^1(\Omega). \tag{2}$$

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By the Poincaré-Friedrichs inequality and the uniform bound on ν, we obtain that the bilinear form in (2) defines an inner product on $H_0^1(\Omega)$. Since $\mathcal{F}$ is a bounded linear functional, the Riesz representation theorem yields well-posedness of (2). For any subdomain $S \subset \Omega$, we define $\mathcal{F}_S(v) := \int_S f v \,\mathrm{d}x$ for all $v \in L^2(S)$. We write $(\cdot,\cdot)_S$ and $\langle\cdot,\cdot\rangle_F$ for the L^2-inner products on subdomains S and faces F, and use $\lesssim$ for inequalities up to constants independent of model parameters and mesh size.

2 Discontinuous Galerkin Method

For ease of presentation, we employ a matching shape-regular simplicial mesh $\mathbb{T}_h$ (cf. [2, Section 1.4.1]) and define $V_h := \mathcal{P}_1^{\mathrm{disc}}(\Omega, \mathbb{T}_h)$. For all $T \in \mathbb{T}_h$, we denote by h_T the diameter of T and define the mesh size $h := \max_{T\in\mathbb{T}_h} h_T$. We assume that the coefficient ν is resolved by the mesh, i.e., ν is piecewise constant with respect to $\mathbb{T}_h$. The sets of interior and boundary faces of $\mathbb{T}_h$ are denoted by $\mathbb{F}_h^i$ and $\mathbb{F}_h^\partial$, respectively. For an interior face $F = \partial T_1 \cap \partial T_2$, we introduce $\nu_i = \nu|_{T_i}$, and we set $\nu_1 = \nu_2 = \nu$ for boundary edges. To deal with varying ν, we introduce ν-dependent weights in the penalty and consistency terms ([2, Chapter 4.5]) and define the sum operators

$$[\![u]\!]_w := \frac{2\nu_2}{\nu_1+\nu_2} u_1 + \frac{2\nu_1}{\nu_1+\nu_2} u_2, \qquad [\![u]\!] := u_1 + u_2, \qquad u \in V_h^d.$$

For any subdomain $D \subset \Omega$, we define $\mathbb{F}_h^i(D) := \{F \cap D : F \in \mathbb{F}_h^i\}$ and $\mathbb{F}_h^\partial(D) := \{F \cap \partial D : F \in \mathbb{F}_h^\partial\}$, and we introduce the bilinear forms

$$\begin{aligned}
B_D^i(u,v) &= B_{p,D}^i(u,v) - B_{c,D}^i(u,v) - B_{c,D}^i(v,u),\\
B_D^\partial(u,v) &= B_{p,D}^\partial(u,v) - B_{c,D}^\partial(u,v) - B_{c,D}^\partial(v,u),\\
B_D(u,v) &= (\nu\nabla_h u, \nabla_h v)_D + B_D^i(u,v) + B_D^\partial(u,v),
\end{aligned}$$

with

$$\begin{aligned}
B_{c,D}^i(u,v) &= \frac{1}{2}\langle [\![\nu\nabla_h u]\!]_w, [\![v\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(D)}, \qquad & B_{p,D}^i(u,v) &= \langle \gamma_h^2 [\![u\boldsymbol{n}]\!], [\![v\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(D)},\\
B_{c,D}^\partial(u,v) &= \langle \nu\nabla_h u, v\boldsymbol{n}\rangle_{\mathbb{F}_h^\partial(D)}, & B_{p,D}^\partial(u,v) &= 2\langle \gamma_h^2 u\boldsymbol{n}, v\boldsymbol{n}\rangle_{\mathbb{F}_h^\partial(D)},
\end{aligned}$$

where $\boldsymbol{n}$ is the elementwise unit outward normal vector, $\langle\cdot,\cdot\rangle_{\mathbb{F}_h^i(D)}$ and $\langle\cdot,\cdot\rangle_{\mathbb{F}_h^\partial(D)}$ denote the L^2-inner products integrated over the union of all faces in $\mathbb{F}_h^i(D)$ and $\mathbb{F}_h^\partial(D)$, respectively. Further, $\gamma_h^2 = \frac{\gamma_0^2}{h_F}\frac{2\nu_1\nu_2}{\nu_1+\nu_2}$ with a stabilization parameter, γ_0, and the face diameter, h_F. Note that $[\![u\boldsymbol{n}]\!]$ describes a jump term by definition.

If $D = \Omega$, we drop the subscript D in the notations above. The discrete formulation of (2) is to find $u^e \in V_h$ such that $B(u^e, v) = \mathcal{F}(v)$ for all $v \in V_h$. For later use, we define D^+ as the union of all $T \in \mathbb{T}_h$ with $T \cap D \neq \emptyset$ and D^- as the union of all $T \in \mathbb{T}_h$ with $\overline{T} \cap D^c = \emptyset$, such that $D^- \subset D \subset D^+$. Throughout this paper, complements are taken with respect to Ω.

3 MS-GFEM for DG Discretizations of Elliptic PDEs

Consider a mesh-resolved overlapping domain decomposition $\{\omega_j\}_{j=1}^M, \cup_{j=1}^M \omega_j = \Omega$, and define a partition of unity $\{\chi_j\}_{j=1}^M$ subordinate to this decomposition satisfying $\mathrm{supp}(\chi_j) \subset \overline{\omega_j^-}, 0 \leq \chi_j \leq 1, \sum_{j=1}^M \chi_j \equiv 1, \chi_j \in \mathcal{P}_1(\Omega, \mathbb{T}_h)$. MS-GFEM builds local approximations on so-called oversampling domains ω_j^* satisfying $\omega_j \subset \omega_j^* \subset \Omega$ as the sum of a local source solution u_j^p with $u_j^p = 0$ on $\partial\omega_j^* \cap \partial\Omega$, cf. (6), and a correction from an optimal n_j-dimensional spectral coarse space $S_{n_j}(\omega_j)$, cf. (7). The global approximation is then assembled using a partition of unity, $u^p := \sum_{j=1}^M \chi_j u_j^p$, $S_n(\Omega) := \left\{ \sum_{j=1}^M \chi_j \phi_j : \phi_j \in S_{n_j}(\omega_j) \right\}$, $n = \sum_{j=1}^M n_j$, and the MS-GFEM approximation is defined as $u^G = u^p + u^s$ with $u^s \in S_n(\Omega)$ satisfying $B(u^s, v) = \mathcal{F}(v) - B(u^p, v)$ for all $v \in S_n(\Omega)$. The key assumptions for the exponential error decay are a Caccioppoli inequality and a weak approximation property. We verify these, together with the remaining assumptions of [3], to show that our method fits into their general framework. For a detailed description of the abstract MS-GFEM and the complete statement of all assumptions, we refer the reader to [3].

Verification of [3, Assumption 2.3]. (i) For $D \subset \Omega$, we define $\mathcal{H}(D) := \{v|_D : v \in V_h\}$ and $\mathcal{H}_0(D) := \{v|_D : v \in V_h, v = 0 \text{ on } D \setminus D^-\}$, both with the inner product

$$(\cdot,\cdot)_{\mathcal{H}(D)} = (\nu\nabla_h\cdot, \nabla_h\cdot)_D + \langle\gamma_h^2[\![\cdot\boldsymbol{n}]\!], [\![\cdot\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(D)} + \langle\gamma_h^2\cdot,\cdot\rangle_{\mathbb{F}_h^\partial(D)} + (\cdot,\cdot)_D.$$

Clearly, $\mathcal{H}_0(D) \subset \mathcal{H}(D)$ for arbitrary subdomains $D \subset \Omega$, and $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$.
(ii) For $D \subset D^*$, we define $E_{D,D^*} : \mathcal{H}_0(D) \to \mathcal{H}_0(D^*)$ via $E_{D,D^*}(v) = v$ on D^- and $E_{D,D^*}(v) = 0$ elsewhere. We want to show that $\|E_{D,D^*}(v)\|_{\mathcal{H}_0(D^*)} = \|v\|_{\mathcal{H}_0(D)}$ for all $v \in \mathcal{H}_0(D)$. Since $E_{D,D^*}(v)$ and v agree on D^- and vanish elsewhere, the volume contributions to the norms are equal. Next, we consider the contributions from interior faces. Let $F \in \mathbb{F}_h^i$ with $F = \partial T_1 \cap \partial T_2$. We want to show

$$\langle\gamma_h^2[\![E_{D,D^*}(v)\boldsymbol{n}]\!], [\![E_{D,D^*}(v)\boldsymbol{n}]\!]\rangle_{F\cap D^*} = \langle\gamma_h^2[\![v\boldsymbol{n}]\!], [\![v\boldsymbol{n}]\!]\rangle_{F\cap D}. \tag{3}$$

If $F \subset D$, then $E_{D,D^*}(v)|_{T_1\cup T_2} = v|_{T_1\cup T_2}$ and $F \cap D = F \cap D^*$, hence (3) holds. If $F \subset D^c$, then $E_{D,D^*}(v)|_{T_1\cup T_2} = 0$ and $F \cap D = \emptyset$, hence both sides of (3) vanish. If $F \cap \partial D \neq \emptyset$, then $E_{D,D^*}(v)|_{T_1\cup T_2} = v|_{T_1\cup T_2} = 0$ and both sides of (3) vanish. We proceed similarly for boundary faces $F \in \mathbb{F}_h^\partial$ with $F \subset \partial T$ to show that

$$\langle\gamma_h^2 E_{D,D^*}(v), E_{D,D^*}(v)\rangle_{F\cap\partial D^*} = \langle\gamma_h^2 v, v\rangle_{F\cap\partial D}. \tag{4}$$

If T is not contained in D^-, then $E_{D,D^*}(v)|_T = 0$ and $v|_T = 0$ and thus both sides of (4) vanish. If T is contained in D^-, then $E_{D,D^*}(v)|_T = v|_T$ and since F is a boundary face with $T \subset D \subset D^*$, we have $F \subset \partial D$ and $F \subset \partial D^*$. Hence, $F \cap \partial D = F = F \cap \partial D^*$ and both sides of (4) are equal.
(iii) For $D \subset D^*$, we define $R_{D^*,D} : \mathcal{H}(D^*) \to \mathcal{H}(D)$, $R_{D^*,D}(v) = v|_D$. Due to the definition of the norm on $\mathcal{H}(D)$, we have $\|R_{D^*,D}(v)\|_{\mathcal{H}(D)} \leq \|v\|_{\mathcal{H}(D^*)}$.

(iv) For arbitrary $D \subset D^*$, $u \in \mathcal{H}(D^*)$ and $v \in \mathcal{H}_0(D)$, we have $B_D(u|_D, v) = B_{D^*}(u, E_{D,D^*}(v))$ since v vanishes on $D \setminus D^-$. □

We denote by $I_h \colon \{v : v|_T \in C^\infty(T) \text{ for all } T \in \mathbb{T}_h\} \to \mathcal{H}(\Omega)$ the elementwise defined Lagrangian interpolation operator and define the partition of unity operator $P_j \colon \mathcal{H}(\omega_j) \to \mathcal{H}_0(\omega_j)$, $P_j(u) = I_h(\chi_j u)$. Boundedness of this operator is shown via the following lemma.

Lemma 1 *Let* $u \in \mathcal{H}(\omega)$, $\chi \in \mathcal{P}_1(\omega, \mathbb{T}_h)$. *Then,* $\|\chi u - I_h(\chi u)\|_{\mathcal{H}_0(\omega)} \lesssim \|\chi u\|_{\mathcal{H}(\omega)}$.

Proof. First, we note that χu is piecewise polynomial due to $\chi \in \mathcal{P}_1(\omega, \mathbb{T}_h)$. Therefore, an inverse inequality and elementwise interpolation properties of I_h yield

$$\begin{aligned} \|\nu^{1/2}\nabla_h(\chi u - I_h(\chi u))\|_{L^2(\omega)} &\lesssim \|\nu^{1/2}\nabla_h(\chi u)\|_{L^2(\omega)}, \\ \|\chi u - I_h(\chi u)\|_{L^2(\omega)} &\lesssim \|\chi u\|_{L^2(\omega)}. \end{aligned}$$

Next, we consider the jump terms. For all interior faces $F = \partial T_1 \cap \partial T_2$, observe that $\langle \llbracket a\boldsymbol{n} \rrbracket, \llbracket b\boldsymbol{n} \rrbracket \rangle_F \le (\|a|_{T_1}\|_{L^2(F)} + \|a|_{T_2}\|_{L^2(F)})(\|b|_{T_1}\|_{L^2(F)} + \|b|_{T_2}\|_{L^2(F)})$ for arbitrary a and b, such that applying the discrete trace inequality [2, Lemma 1.46], elementwise interpolation properties and [2, Lemma 1.43] yields

$$\begin{aligned} &\langle \gamma_h^2 \llbracket (\chi u - I_h(\chi u))\boldsymbol{n} \rrbracket, \llbracket (\chi u - I_h(\chi u))\boldsymbol{n} \rrbracket \rangle_F \\ &\lesssim \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} h_{T_{\min}}^{-2} \|\chi u - I_h(\chi u)\|_{L^2(T_1 \cup T_2)}^2 \lesssim \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} h_{T_{\min}}^{-2} h_{T_{\max}}^2 \|\nabla_h(\chi u)\|_{L^2(T_1 \cup T_2)}^2 \\ &\lesssim \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \sum_{i=1}^{2} \frac{1}{\nu_i} \|\nu_i^{1/2} \nabla_h(\chi u)\|_{L^2(T_i)}^2 \le \|\nu^{1/2}\nabla_h(\chi u)\|_{L^2(T_1 \cup T_2)}^2, \end{aligned} \tag{5}$$

where $h_{T_{\min}} = \min\{h_{T_1}, h_{T_2}\}$ and $h_{T_{\max}} = \max\{h_{T_1}, h_{T_2}\}$. Summing over all interior faces of ω, we conclude

$$\langle \gamma_h^2 \llbracket (\chi u - I_h(\chi u))\boldsymbol{n} \rrbracket, \llbracket (\chi u - I_h(\chi u))\boldsymbol{n} \rrbracket \rangle_{\mathbb{F}_h^i(\omega)} \lesssim \|\nu^{1/2}\nabla_h(\chi u)\|_{L^2(\omega)}^2.$$

Boundary faces can be treated similarly. Finally, we combine all estimates to obtain the statement of the lemma. □

Corollary 1 *Let* $u \in \mathcal{H}(\omega)$ *and* $\chi \in \mathcal{P}_1(\omega, \mathbb{T}_h)$. *Then,*

$$\|I_h(\chi u)\|_{\mathcal{H}_0(\omega)} \lesssim \|\chi u\|_{\mathcal{H}(\omega)} \lesssim \sqrt{1 + \|\nabla_h \chi\|_{L^\infty(\omega)}^2}\, \|u\|_{\mathcal{H}(\omega)}.$$

Verification of [3, Assumption 2.9]. Let $\psi_j \in \mathcal{H}_0(\omega_j^*)$ be the unique solution of

$$B_{\omega_j^*}(\psi_j, v) = \mathcal{F}_{\omega_j^*}(v) \qquad \forall v \in \mathcal{H}_0(\omega_j^*) \tag{6}$$

and define the particular solution as $u_j^p := \psi_j|_{\omega_j}$. Note that $B_{\omega_j^*}$ does not contain face integrals over the interior subdomain boundary. However, since we define problem (6) on $\mathcal{H}_0(\omega_j^*)$, it is well-posed. □

Verification of [3, Assumption 2.13]. For every domain $\omega_j \subset D \subset \omega_j^*$, we define $B_D^+(\cdot,\cdot) := (\nu\nabla_h\cdot, \nabla_h\cdot)_D + \langle\gamma_h^2[\![\cdot\boldsymbol{n}]\!], [\![\cdot\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(D)} + \langle\gamma_h^2[\![\cdot\boldsymbol{n}]\!], [\![\cdot\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^\partial(D)}$. If $B_D^+(u,v) = 0$ holds for all $v \in \mathcal{H}(D)$, then this holds in particular for $v = u$. Thus, the gradient terms and jumps over inner faces in the definition of B_D^+ imply that u is constant on D. Conversely, if u is constant, the jump terms and gradients in the definition of B_D^+ vanish and $B_D^+(u,v) = 0$ holds for all $v \in \mathcal{H}(D)$. Hence, [3, Assumption 2.13] holds with $\mathcal{K}_D = \{0\}$ if D is a boundary subdomain and with $\mathcal{K}_D = \operatorname{span}(1)$ if D is an interior subdomain. □

For the construction of the coarse space, we define the space of discretely locally harmonic functions, $\mathcal{H}_B(\omega_j^*) := \{u \in \mathcal{H}(\omega_j^*) : B_{\omega_j^*}(u,v) = 0 \ \forall\ v \in \mathcal{H}_0(\omega_j^*)\}$. Consider the generalized eigenvalue problem of finding $\lambda \in [0,+\infty]$, $\varphi \in \mathcal{H}_B(\omega_j^*)$ such that $B_{\omega_j}^+(P_j(\varphi|_{\omega_j}), P_j(v|_{\omega_j})) = \lambda B_{\omega_j^*}^+(\varphi, v)$ for all $v \in \mathcal{H}_B(\omega_j^*)$. Denoting the k-th eigenpair as $(\lambda_{j,k}, \varphi_{j,k})$, where $\lambda_{j,1} \geq \lambda_{j,2} \geq \ldots$, the local approximation space is built from the eigenfunctions corresponding to the n_j largest eigenvalues:

$$S_{n_j}(\omega_j) := \operatorname{span}\{\varphi_{j,1}|_{\omega_j}, \ldots, \varphi_{j,n_j}|_{\omega_j}\}. \tag{7}$$

Next, we verify the two central assumptions of [3]. For $D \subset \Omega$, we denote by $C_B(D)$ and $\alpha_B(D)$ the continuity and coercivity constants of the bilinear form B_D with respect to the (semi-)norm $\|\cdot\|_{B^+,D}$ induced by B_D^+. Note that $C_B(D)$ and $\alpha_B(D)$ are independent of ν, which can be shown as in [2, Lemma 4.51, Lemma 4.52].

Lemma 2 (Caccioppoli inequality, [3, Assumption 3.1]) *Let* $\omega \subset \omega^* \subset \Omega$ *and* $u \in \mathcal{H}_B(\omega^*)$ *with* $\delta = \operatorname{dist}(\omega, \partial\omega^* \setminus \partial\Omega) > 3 \max_{K \in \mathbb{T}_h : K \cap \omega^* \setminus \omega \neq \emptyset} h_K$. *Then,*

$$\|u|_\omega\|_{B^+,\omega} \lesssim \nu_{\max}^{1/2}\delta^{-1}\|u\|_{L^2(\omega^*\setminus\omega)}. \tag{8}$$

Proof. Let $\eta \in \mathcal{P}_1(\omega^*, \mathbb{T}_h)$ be a cut-off function with $\operatorname{supp}(\eta) \subset \overline{(\omega^*)^-}$, $\eta = 1$ on ω^+ and $|\nabla_h\eta| \leq C_\eta\delta^{-1}$. We proceed as in [4]. By definition,

$$B_{\omega^*}(\eta u, \eta u) = (\nu\nabla_h(\eta u), \nabla_h(\eta u))_{\omega^*} + B_{\omega^*}^i(\eta u, \eta u) + B_{\omega^*}^\partial(\eta u, \eta u). \tag{9}$$

For the second term in (9), we use that η is continuous along mesh faces to compute

$$\begin{aligned}
B_{\omega^*}^i(\eta u, \eta u) &= \langle\gamma_h^2[\![\eta u\boldsymbol{n}]\!], [\![\eta u\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(\omega^*)} - \langle[\![\nu\nabla_h(\eta u)]\!]_w, [\![(\eta u)\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(\omega^*)} \\
&= \langle\gamma_h^2[\![u\boldsymbol{n}]\!], [\![\eta^2 u\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(\omega^*)} - \langle[\![\nu\eta\nabla_h u + \nu u\nabla_h\eta]\!]_w, [\![(\eta u)\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(\omega^*)} \\
&= B_{p,\omega^*}^i(u, \eta^2 u) - B_{c,\omega^*}^i(u, \eta^2 u) - \langle[\![\tfrac{1}{2}\nu\eta\nabla_h u + \nu u\nabla_h\eta]\!]_w, [\![(\eta u)\boldsymbol{n}]\!]\rangle_{\mathbb{F}_h^i(\omega^*)} \\
&= B_{p,\omega^*}^i(u, \eta^2 u) - B_{c,\omega^*}^i(u, \eta^2 u) - B_{c,\omega^*}^i(\eta^2 u, u) = B_{\omega^*}^i(u, \eta^2 u).
\end{aligned}$$

With the same arguments, we obtain $B_{\omega^*}^\partial(\eta u, \eta u) = B_{\omega^*}^\partial(u, \eta^2 u)$. Since η is supported on $(\omega^*)^-$, the harmonicity of u implies $B_{\omega^*}(u, I_h(\eta^2 u)) = 0$, which we subtract from (9) while plugging in the above identities and using the product rule to obtain

$$\begin{aligned} B_{\omega^*}(\eta u, \eta u) = (\nu u \nabla_h \eta, u \nabla_h \eta)_{\omega^*} + (\nu \nabla_h u, \nabla_h(\eta^2 u - I_h(\eta^2 u)))_{\omega^*} \\ + B^i_{\omega^*}(u, \eta^2 u - I_h(\eta^2 u)) + B^\partial_{\omega^*}(u, \eta^2 u - I_h(\eta^2 u)). \end{aligned} \tag{10}$$

We will now show that

$$B_{\omega^*}(\eta u, \eta u) \leq C \nu_{\max}^{1/2} \|u\|_{L^2(\omega^* \setminus \omega)} \left(\frac{1}{\delta} \|\nu^{1/2} \nabla_h(\eta u)\|_{L^2(\omega^* \setminus \omega)} + \frac{\nu_{\max}^{1/2}}{\delta^2} \|u\|_{L^2(\omega^* \setminus \omega)} \right).$$

Once we have this inequality, we can use a weighted Young's inequality to absorb the term involving the gradient of ηu and infer

$$B_{\omega^*}(\eta u, \eta u) \leq \left(C + \frac{C^2}{2\alpha_B(\Omega)} \right) \frac{\nu_{\max}}{\delta^2} \|u\|^2_{L^2(\omega^* \setminus \omega)} + \frac{\alpha_B(\Omega)}{2} \|\nu^{1/2} \nabla_h(\eta u)\|^2_{L^2(\omega^* \setminus \omega)}. \tag{11}$$

Since η is supported on $(\omega^*)^-$, we have $B(\eta u, \eta u) = B_{\omega^*}(\eta u, \eta u)$ and thus $\alpha_B(\Omega) \|\eta u\|^2_{B^+, \Omega} \leq B_{\omega^*}(\eta u, \eta u)$ due to the coercivity of B. Combining this with (11), the assumptions on η, and noting that the face integrals in the definition of B^+_ω are bounded by the face integrals in the definition of B^+_Ω establishes the lemma.

Now, we bound all terms in (10). The first term can be bounded from above by $(\nu u \nabla_h \eta, u \nabla_h \eta)_{\omega^*} \leq \nu_{\max} C_\eta^2 \delta^{-2} \|u\|^2_{L^2(\omega^* \setminus \omega)}$. Since $\eta|_{\omega^+} = 1$, we infer that all terms in (10) only have contributions from elements contained in $(\omega^*)^- \setminus \omega^+ \neq \emptyset$. Let $T \in \mathbb{T}_h$ be such an element. Using [3, Lemma 5.4] and an inverse inequality [2, Lemma 1.44], we obtain

$$\begin{aligned} (\nu \nabla_h u, \nabla_h(\eta^2 u - I_h(\eta^2 u)))_T &\leq \nu|_T \|\nabla_h u\|_{L^2(T)} \|\nabla_h(\eta^2 u - I_h(\eta^2 u))\|_{L^2(T)} \\ &\lesssim \nu|_T \|\nabla_h u\|_{L^2(T)} \left(\frac{h_T}{\delta} \|\nabla_h(\eta u)\|_{L^2(T)} + \frac{h_T}{\delta^2} \|u\|_{L^2(T)} \right) \\ &\lesssim \nu_{\max}^{1/2} \|u\|_{L^2(T)} \left(\frac{1}{\delta} \|\nu^{1/2} \nabla_h(\eta u)\|_{L^2(T)} + \frac{\nu_{\max}^{1/2}}{\delta^2} \|u\|_{L^2(T)} \right) \end{aligned}$$

and hence

$$\begin{aligned} &(\nu \nabla_h u, \nabla_h(\eta^2 u - I_h(\eta^2 u)))_{\omega^*} \\ &\lesssim \nu_{\max}^{1/2} \|u\|_{L^2(\omega^* \setminus \omega)} \left(\frac{1}{\delta} \|\nu^{1/2} \nabla_h(\eta u)\|_{L^2(\omega^* \setminus \omega)} + \frac{\nu_{\max}^{1/2}}{\delta^2} \|u\|_{L^2(\omega^* \setminus \omega)} \right). \end{aligned} \tag{12}$$

It remains to bound the last two terms of (10). We only consider interior faces since the boundary terms can be treated in the same way. Recall that by definition $B^i_{\omega^*}(u, v) = B^i_{p,\omega^*}(u, v) - B^i_{c,\omega^*}(u, v) - B^i_{c,\omega^*}(v, u)$. Consider two adjacent elements $T_1, T_2 \in \mathbb{T}_h$ with common face $F \in \mathbb{F}^i_h$ and abbreviate $T := T_1 \cup T_2$ and $\nu_i = \nu|_{T_i}$. We bound the penalty term B^i_{p,ω^*} face by face:

$$\begin{aligned}
\langle \gamma_h^2 [\![u\boldsymbol{n}]\!], [\![(\eta^2 u - I_h(\eta^2 u))\boldsymbol{n}]\!] \rangle_F &\lesssim \frac{\nu_1\nu_2}{\nu_1+\nu_2} h_{T_{\min}}^{-2} \|u\|_{L^2(T)} \|\eta^2 u - I_h(\eta^2 u)\|_{L^2(T)} \\
&\lesssim \frac{\nu_1\nu_2}{\nu_1+\nu_2} h_{T_{\min}}^{-2} \|u\|_{L^2(T)} \left(\frac{h_{T_{\max}}^2}{\delta} \|\nu^{1/2}\nabla_h(\eta u)\|_{L^2(T)} + \frac{h_{T_{\max}}^2}{\delta^2} \|u\|_{L^2(T)} \right) \\
&\lesssim \nu_{\max}^{1/2} \|u\|_{L^2(T)} \left(\frac{1}{\delta} \|\nu^{1/2}\nabla_h(\eta u)\|_{L^2(T)} + \frac{\nu_{\max}^{1/2}}{\delta^2} \|u\|_{L^2(T)} \right),
\end{aligned} \tag{13}$$

where the first step is carried out analogously to the first step in (5), and we further used [3, Lemma 5.4] as well as [2, Lemma 1.43]. Note that the condition $\eta \in \mathcal{P}_1(\omega^*, \mathbb{T}_h)$ is crucial for the applicability of the discrete trace inequality, since it guarantees that $\eta^2 u$ is piecewise polynomial. Summing over all faces, we obtain the same upper bound for $B^i_{p,\omega^*}(u, \eta^2 u - I_h(\eta^2 u))$ as in (12).

For the first consistency term $B^i_{c,\omega^*}(u, \eta^2 u - I_h(\eta^2 u))$ of $B^i_{\omega^*}$, we again proceed as in (5) to obtain

$$\begin{aligned}
\langle [\![\nu\nabla_h u]\!]_w, [\![\eta^2 u - I_h(\eta^2 u)\boldsymbol{n}]\!] \rangle_F &\lesssim \frac{\nu_1\nu_2}{\nu_1+\nu_2} h_{T_{\min}}^{-1} \|\nabla_h u\|_{L^2(T)} \|\eta^2 u - I_h(\eta^2 u)\|_{L^2(T)} \\
&\lesssim \frac{\nu_1\nu_2}{\nu_1+\nu_2} h_{T_{\min}}^{-2} \|u\|_{L^2(T)} \|\eta^2 u - I_h(\eta^2 u)\|_{L^2(T)},
\end{aligned}$$

which can then be treated in the same way as (13), such that summing over all faces once again yields the same upper bound for $B^i_{c,\omega^*}(u, \eta^2 u - I_h(\eta^2 u))$ as in (12). The term $B^i_{c,\omega^*}(\eta^2 u - I_h(\eta^2 u), u)$ can be treated analogously. □

Remark 1 Assuming that ν is piecewise constant is not strictly necessary but leads to smaller constants in Corollary 1 and Lemma 2. For general diffusion coefficients, an additional factor $\sqrt{\nu_{\max}/\nu_{\min}}$ occurs in the estimates, cf. [4, Lemma 3.10].

Lemma 3 (Weak approximation property, [3, Assumption 3.4]) *Let $\omega \subset \omega^* \subset \omega^{**}$ be subdomains of Ω with $\delta := \operatorname{dist}(\omega^*, \partial\omega^{**} \setminus \partial\Omega) > \max_{K \in \mathbb{T}_h : K \cap \omega^* \setminus \omega \neq \emptyset} h_K$, and let $\mathsf{V}_\delta(\omega^* \setminus \omega) := \{ \boldsymbol{x} \in \omega^{**} : \operatorname{dist}(\boldsymbol{x}, \omega^* \setminus \omega) \le \delta \}$. Then, there exists a constant $C_1 > 0$ depending only on d, such that for each integer $m \ge C_1 |\mathsf{V}_\delta(\omega^* \setminus \omega)| \delta^{-d}$ and $h \le |\mathsf{V}_\delta(\omega^* \setminus \omega)|^{1/d} m^{-1/d}$, there exists an m-dimensional space $Q_m(\omega^{**}) \subset L^2(\omega^{**})$ such that for all $u \in \mathcal{H}_B(\omega^{**})$,*

$$\inf_{v \in Q_m(\omega^{**})} \|u - v\|_{L^2(\omega^* \setminus \omega)} \lesssim \nu_{\min}^{-1/2} |\mathsf{V}_\delta(\omega^* \setminus \omega)|^{1/d} m^{-1/d} \|u\|_{B^+, \omega^{**}}.$$

Proof. By the assumption on the mesh size, we have $\delta^- := \operatorname{dist}(\omega^*, \partial(\omega^{**})^- \setminus \partial\Omega) > 0$. We define the set $\mathsf{V}_{\delta^-}(\omega^* \setminus \omega) := \{ \boldsymbol{x} \in (\omega^{**})^- : \operatorname{dist}(\boldsymbol{x}, \omega^* \setminus \omega) \le \delta^- \}$ and denote by $R_h \colon V_h((\omega^{**})^-) \to W^{1,\infty}((\omega^{**})^-)$ the reconstruction operator introduced in [1, Section 3] defined on $(\omega^{**})^-$. By [3, Lemma 5.5], there is an m-dimensional space $Y_m \subset L^2((\omega^{**})^-)$ and a constant $C > 0$ depending only on d such that

$$\inf_{v \in Y_m} \|R_h u - v\|_{L^2(\omega^* \setminus \omega)} \le C |\mathsf{V}_{\delta^-}(\omega^* \setminus \omega)|^{1/d} m^{-1/d} \|\nabla_h R_h u\|_{L^2((\omega^{**})^-)}$$

for all $u \in \mathcal{H}((\omega^{**})^-)$. Using the triangle inequality, the previous estimate and [1, Theorem 3.1], we obtain

$$\begin{aligned}\inf_{v \in Y_m} \|u - v\|_{L^2(\omega^* \setminus \omega)} &\leq \|R_h u - u\|_{L^2(\omega^* \setminus \omega)} + \inf_{v \in Y_m} \|R_h u - v\|_{L^2(\omega^* \setminus \omega)} \\ &\lesssim h\|u\|_{B^+,(\omega^{**})^-} + C_1 \left|\mathsf{V}_{\delta^-}(\omega^* \setminus \omega)\right|^{1/d} m^{-1/d} \|\nabla_h R_h u\|_{L^2((\omega^{**})^-)} \\ &\lesssim \left|\mathsf{V}_{\delta^-}(\omega^* \setminus \omega)\right|^{1/d} m^{-1/d} \nu_{\min}^{-1/2} \|u\|_{B^+,(\omega^{**})^-}.\end{aligned}$$

Defining $Q_m(\omega^{**})$ via extension by zero of functions in Y_m and noting that $\left|\mathsf{V}_{\delta^-}(\omega^* \setminus \omega)\right| \leq \left|\mathsf{V}_{\delta}(\omega^* \setminus \omega)\right|$, the lemma is established. □

Having verified all relevant assumptions of [3], we obtain the desired nearly exponential decay of the eigenvalues and the global MS-GFEM error. Note that the Kolmogorov n-widths given in [3, Theorems 3.8 and 2.23] coincide with the eigenvalues stated below due to [3, Lemmas 2.19 and 2.20].

Theorem 1 ([3, Theorem 3.8]) *Assume that $\omega_j \subset \omega_j^*$ are truncated concentric cubes of side length H_j and H_j^*, respectively, that are resolved by the mesh. Let $H_j^* > H_j$. Then, there exist constants $N_j \in \mathbb{N}$, $C_j > 0$ and $c_j > 0$, independent of h, such that for all $n \geq N_j$, if h is sufficiently small, we have*

$$\sqrt{\lambda_{j,n}} \leq C_j e^{-c_j n^{1/d}}.$$

Theorem 2 *The MS-GFEM solution u^{G} satisfies*

$$\|u^e - u^{\mathrm{G}}\|_{B^+,\Omega} \leq \frac{C_B(\Omega)}{\alpha_B(\Omega)} \sqrt{\kappa\kappa^*} \left(\max_{j=1,\ldots,M} \sqrt{\lambda_{j,n_{j+1}}} \frac{C_B(\omega_j^*)}{\alpha_B(\omega_j^*)} \right) \|u^e\|_{B^+,\Omega},$$

where κ and κ^* are the coloring constants of $\{\omega_j\}_{j=1}^M$ and $\{\omega_j^*\}_{j=1}^M$, respectively.

Proof. This follows from combining Céa's Lemma and [3, Theorem 2.8]. The latter is applicable with $\varepsilon_j = \sqrt{\lambda_{j,n_{j+1}}} C_B(\omega_j^*)/\alpha_B(\omega_j^*) \|u^e\|_{B^+,\omega_j^*}$ due to [3, Theorem 2.23], since $\|u^e|_{\omega_j^*} - \psi_j\|_{B^+,\omega_j^*} \leq C_B(\omega_j^*)/\alpha_B(\omega_j^*) \|u^e\|_{B^+,\omega_j^*}$ and $\|\cdot\|_{B^+,\Omega} \leq \|\cdot\|_{\mathcal{H}(\Omega)}$. □

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