A GenEO-Type Coarse Space with Smaller Eigenproblems

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1 Introduction and problem setting

Coarse spaces are essential to ensure robustness w.r.t. the number of subdomains in two-level overlapping Schwarz methods. Robustness with respect to the coefficients of the underlying partial differential equation (PDE) can be achieved by adaptive (or spectral) coarse spaces involving local eigenproblems [7, 6, 3]. The solution of these eigenproblems, although scalable, entails a large setup cost which may exceed the cost for the iteration phase. Following an idea from [1] we present and analyse a new variant of the GenEO (Generalised Eigenproblems in the Overlap) coarse space [7] which solves eigenproblems only in a strip connected to the boundary of the subdomain. If the overlap parameter is not too large the setup cost is significantly reduced. The method satisfies a coefficient-robust condition number estimate similar to that of the original method, but the size of the coarse space might be increased, see the examples presented in [6, 1]. We call the new method R-GenEO as the domain of the eigenproblems has ring-shape in two dimensions.

In the rest of this section we introduce the problem setting and review the original GenEO coarse space. In Section 2 we introduce the new R-GenEO coarse space and prove its robustness. In the last section we give a brief numerical comparison of both methods.

1.1 Preliminaries

We consider a finite element discretisation of a partial differential equation that is posed on some domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Let V_h be a finite-dimensional Hilbert

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space (the finite element space on some mesh $\mathcal{T}_h$), $a : V_h \times V_h \to \mathbb{R}$ a continuous, symmetric and coercive bilinear form and $f \in V_h'$ a linear form. We assume that a is of the form

$$a(u, v) = \sum_{\tau \in \mathcal{T}_h} a_\tau(u_{|\tau}, v_{|\tau}) , \qquad \text{for all } u, v \in V_h ,$$

for some positive semidefinite bilinear forms $a_\tau : V_h(\tau) \times V_h(\tau) \to \mathbb{R}$, for all $\tau \in \mathcal{T}_h$, where $V_h(\tau) = \{v_{|\tau} \mid v \in V_h\}$. The problem then reads: find $u \in V_h$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in V_h$. While the method is in principle more general, we restrict ourselves to a scalar, second-order elliptic problem where $a(u, v) = \int_\Omega (\alpha(x)\nabla u) \cdot \nabla v \,\mathrm{d}x$. By choosing a basis $\{\varphi_k\}_{k=1,\dots,n}$ of V_h we can write this problem as a linear system $\mathbf{Au} = \mathbf{f}$. Our goal is to solve this linear system using a Krylov method preconditioned by a two-level Schwarz method.

To this end, let $\{\Omega_j\}_{j=1}^N$ be an overlapping domain decomposition of the domain Ω where we assume that the subdomains are resolved by the mesh $\mathcal{T}_h$, and introduce the spaces $V_{h,0}(\Omega_j) := \{v_{|\Omega_j} \mid v \in V_h , \operatorname{supp}(v) \subset \Omega_j\}$. We denote by $R_j^\top : V_{h,0}(\Omega_j) \to V_h$ the extension-by-zero operator. Its adjoint $R_j : V_h' \to V_{h,0}(\Omega_j)'$ is called the restriction operator. In addition we introduce a so-called coarse space $V_H \subset V_h$ which will be defined later. $R_H^\top : V_H \to V_h$ denotes the canonical inclusion mapping each coarse function to the same function viewed as an element of V_h, and R_H its adjoint. If we let $\mathbf{R}_j$, $j = 1, \dots, N$, be the matrix representations of R_j (w.r.t. the basis $\{\varphi_k\}_k$) and $\mathbf{R}_H$ be the matrix representation of R_H (w.r.t. a basis of the coarse space V_H) then the additive two-level Schwarz preconditioner reads

$$\mathbf{M}_{\mathrm{AS},2}^{-1} = \mathbf{R}_H^T \mathbf{A}_H^{-1} \mathbf{R}_H + \sum_{j=1}^N \mathbf{R}_j^T \mathbf{A}_j^{-1} \mathbf{R}_j ,$$

where the subdomain matrices $\mathbf{A}_j$ and the coarse matrix $\mathbf{A}_H$ are defined via Galerkin projection, i.e., $\mathbf{A}_j = \mathbf{R}_j \mathbf{A} \mathbf{R}_j^T$ and $\mathbf{A}_H = \mathbf{R}_H \mathbf{A} \mathbf{R}_H^T$.

Let us recall some standard definitions and results. We denote by $k_0 \in \mathbb{N}$ the maximum number of subdomains that a mesh element belongs to, i.e.,

$$k_0 = \max_{\tau \in \mathcal{T}_h} \left(\# \left\{ \Omega_j \mid j \in \{1, \dots, N\}, \tau \in \Omega_j \right\} \right) .$$

For any set D that is the union of elements of $\mathcal{T}_h$ we let $V_h(D) := \{v_{|D} \mid v \in V_h\}$ and write

$$a_D(u, v) := \sum_{\tau \in D} a_\tau(u_{|\tau}, v_{|\tau}) \qquad \text{for } u, v \in V_h(D) .$$

Note that for $u, v \in V_{h,0}(\Omega_j)$ we have $a_{\Omega_j}(u, v) = a(R_j^\top u, R_j^\top v)$ and this bilinear form is positive definite (on $V_{h,0}(\Omega_j)$). We denote by $|\cdot|_{a,D}$ the seminorm induced by the bilinear form $a_D(\cdot, \cdot)$, and by $\|\cdot\|_{a,\Omega_j}$ the norm induced by $a_{\Omega_j}(\cdot, \cdot)$ on $V_{h,0}(\Omega_j)$. If $D = \Omega$, we omit the domain in the subscript.

Definition 1 (Stable decomposition) Let $C_0 > 0$ be a constant. A C_0-stable decomposition of $u \in V_h$ is a family of functions $\{z_j\}_{j=0,\dots,N}$ such that

$$u = \sum_{j=0}^{N} z_j\,, \qquad \text{with } z_0 \in V_H \text{ and } z_j \in V_{h,0}(\Omega_j), \text{ for } j > 0\,, \tag{1}$$

and

$$\|z_0\|_a^2 + \sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \le C_0^2 \|u\|_a^2\,. \tag{2}$$

Proving existence of a stable decomposition is the key step in deriving condition number bounds for additive Schwarz methods. The following result appears in some form or another in many publications on additive Schwarz methods.

Theorem 1 ([7, Thm. 2.8]) *If every $u \in V_h$ admits a C_0-stable decomposition then* $\kappa(\mathbf{M}_{\mathrm{AS},2}^{-1}\mathbf{A}) \le C_0^2(k_0+1)$.

At the expense of introducing a quadratic dependency on k_0 in the condition number bound, one can show (2) without having to bound $\|z_0\|_a$ in terms of $\|u\|_a$.

Lemma 1 ([7, Lem. 2.9]) *Using the notation of Definition 1, if there exists $C_1 > 0$ s.t.*

$$\|z_j\|_{a,\Omega_j}^2 \le C_1 |u|_{a,\Omega_j}^2\,, \qquad \textit{for all } j = 1, \dots, N\,,$$

then the decomposition (1) *is C_0-stable with $C_0^2 = 2 + C_1 k_0(2k_0 + 1)$.*

1.2 The GenEO coarse space

Let us now briefly recall the definition of the original GenEO coarse space introduced in [7]. To this end, we first define the overlapping zone (or just the overlap, for short) of subdomain Ω_j, $j = 1, \dots, N$, as

$$\Omega_j^\circ = \left\{ x \in \Omega_j \mid x \in \Omega_{j'} \text{ for some } j' \ne j \right\}\,.$$

Next, let $\xi_j : V_h(\Omega_j) \to V_{h,0}(\Omega_j)$ be partition of unity operators that satisfy

$$\sum_{j=1}^{N} R_j^\top \xi_j(v_{|\Omega_j}) = v\,, \qquad \text{for all } v \in V_h\,,$$

and $\xi_j(v)_{|\Omega_j \setminus \Omega_j^\circ} = v_{|\Omega_j \setminus \Omega_j^\circ}$, for all $v \in V_h(\Omega_j)$ and $j = 1, \dots, N$. On each overlapping subdomain Ω_j, $j = 1, \dots, N$, we now consider the generalised eigenvalue problem: Find $(\lambda^j, t^j) \in \mathbb{R} \times V_h(\Omega_j)$ such that

$$a_{\Omega_j}(u, t^j) = \lambda^j a_{\Omega_j^\circ}(\xi_j(u), \xi_j(t^j))\,, \qquad \text{for all } u \in V_h(\Omega_j). \tag{3}$$

For $j = 1, \dots, N$ let $\{t_j^k\}_{k=1,\dots,m_j}$ be the eigenfunctions of the eigenproblem (3) corresponding to the m_j smallest eigenvalues. The GenEO coarse space is defined as

$$V_H := \operatorname{span}\left\{R_j^\top \xi_j(t_k^j) \mid k = 1, \ldots, m_j\,; j = 1, \ldots, N\right\}\,.$$

In [7] the authors prove that the condition number of the two-level Schwarz method with the GenEO coarse space can be bounded by

$$\kappa(\mathbf{M}_{\mathrm{AS},2}^{-1}\mathbf{A}) \le (1 + k_0)(2 + k_0(2k_0 + 1) \max_{1\le j\le N}(1 + 1/\lambda_{m_j+1}^j))\,.$$

2 The R-GenEO coarse space

To define the new coarse space we introduce some additional sets and notation. Let $\Omega_j^* \subseteq \Omega_j$ be a set that satisfies $\Omega_j^\circ \subset \Omega_j^*$ (e.g. Ω_j^* can be obtained by extending Ω_j° by one layer of grid elements towards the interior of Ω_j and this is what will be used in the numerical experiments). We further define

$$\omega_j^\circ := \overline{\Omega_j} \setminus \overline{\Omega_j^\circ}\,, \quad \Gamma_j^\circ := \partial\Omega_j^\circ \cap \partial\omega_j^\circ\,, \quad \omega_j^* := \overline{\Omega_j} \setminus \overline{\Omega_j^*}\,, \quad \Gamma_j^* := \partial\Omega_j^* \cap \partial\omega_j^*\,,$$

and we assume that there exists $\delta > 0$ such that $\operatorname{dist}(\Gamma_j^*, \Gamma_j^\circ) \ge \delta$. Now consider the following eigenvalue problem on each overlapping subdomain: Find $(\lambda^j, t^j) \in \mathbb{R} \times V_h(\Omega_j^*)$ such that

$$a_{\Omega_j^*}(u, t^j) = \lambda^j a_{\Omega_j^*}(\eta_j(u), \eta_j(t^j))\,, \qquad \text{for all } u \in V_h(\Omega_j^*), \tag{4}$$

where $\eta_j : V_h(\Omega_j^*) \to V_{h,0}(\Omega_j^*)$ are functions that satisfy $(\eta_j v)(x) = (\xi_j v)(x)$ for $x \in \overline{\Omega_j^\circ}$, in particular $(\eta_j v)(x) = 0$ for $x \in \Gamma_j^*$ and $(\eta_j v)(x) = 1$ for $x \in \Gamma_j^\circ$. Let t_k^j, $k = 1, \ldots, m_j$, denote the eigenvectors corresponding to the m_j smallest eigenvalues of (4). To obtain the local components of the basis vectors y_k^j that make up the coarse space we proceed as follows: on $\overline{\Omega_j^\circ}$ we take $y_k^j = t_k^j$. On ω_j° we compute y_k^j as an operator-harmonic extension of $t_k^j|_{\Gamma_j^\circ}$. More precisely, consider the a-orthogonal projection operator $Q_j^\circ : V_h(\omega_j^\circ) \to V_{h,0}(\omega_j^\circ)$ which is defined via

$$a_{\omega_j^\circ}(Q_j^\circ v, z) = a_{\omega_j^\circ}(v, z) \qquad \text{for all } z \in V_{h,0}(\omega_j^\circ)\,,$$

and set $H_j^\circ = I - Q_j^\circ$. To define the harmonic extension of the eigenfunctions (which are elements of $V_h(\Omega_j^*)$) we introduce the trace operator γ_j° defined by $\gamma_j^\circ v = v|_{\Gamma_j^\circ}$ and an extension operator $E_j^\circ : V_h(\Gamma_j^\circ) \to V_h(\omega_j^\circ)$ that satisfies $\gamma_j^\circ E_j^\circ = I$ on $V_h(\Gamma_j^\circ)$ (i.e., E_j° takes a finite element function on the boundary Γ_j° and extends it arbitrarily to the interior domain ω_j°). The local components of the coarse space vectors are then defined as

$$y_k^j(x) = \begin{cases} t_k^j(x)\,, & \text{for } x \in \overline{\Omega_j^\circ}, \\ H_j^\circ E_j^\circ \gamma_j^\circ t_k^j(x)\,, & \text{for } x \in \omega_j^\circ, \end{cases} \tag{5}$$

for $k = 1, \dots, m_j$. We introduce the short notation $y_k^j = \mathcal{H}_{\Omega_j^\circ \to \Omega_j}(t_k^j)$ for the mapping defined by (5). In practice, the second line of (5) amounts to discarding the values of t_k^j in $\Omega_j^* \cap \omega_j^\circ$ and computing the harmonic extension from Γ_j° to ω_j°.

Definition 2 (R-GenEO coarse space) For $j = 1, \dots, N$ let $\{t_j^k\}_{k=1,\dots,m_j}$ be the eigenfunctions of the eigenproblem (4) corresponding to the m_j smallest eigenvalues. The R-GenEO coarse space is defined as

$$V_H := \operatorname{span}\left\{ R_j^\top \xi_j (\mathcal{H}_{\Omega_j^\circ \to \Omega_j}(t_k^j)) \mid k = 1, \dots, m_j\,; j = 1, \dots, N \right\} .$$

Remark 1 As mentioned above, a similar construction was recently used in the context of a multiscale generalised finite element method [1]. The idea of computing the coarse basis vectors using energy-minimising extensions of eigenvectors is also present in the so called adaptive GDSW coarse spaces, see, e.g., [6].

To prove robustness of the coarse space, we make use of the following result which was shown in [3].

Lemma 2 ([3, Sec. 3.3]) *Let V be a n-dimensional vector space and let $b, c : V \times V \to \mathbb{R}$ be two positive semidefinite bilinear forms on V with* $\ker b \cap \ker c = \{0\}$. *Consider the generalised eigenvalue problem: Find $(\lambda, p) \in (\mathbb{R} \cup \{\infty\}) \times V$, $p \neq 0$, such that either $p \notin \ker c$ and*

$$b(p, v) = \lambda c(p, v) \qquad \text{for all } v \in V$$

or $p \in \ker c$ and $\lambda = \infty$. Let the eigenpairs $\{(p_k, \lambda_k)\}_{k=1}^n$ of this problem be ordered such that $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \infty$. Suppose that $m \in \{1, \dots, n\}$ is such that $0 < \lambda_{m+1} < \infty$. Then, the projection operator

$$\Pi_m v := \sum_{k=1}^{m} c(v, p_k) p_k$$

is well-defined and orthogonal w.r.t. the bilinear form $b(\cdot, \cdot)$. Thus $|\Pi_m v|_b \leq |v|_b$ and $|v - \Pi_m v|_b \leq |v|_b$ and we have the stability estimate

$$|v - \Pi_m v|_c^2 \leq 1/\lambda_{m+1} |v - \Pi_m v|_b^2 \qquad \text{for all } v \in V.$$

To apply the lemma, we have to show that the kernels of the bilinear forms that appear in the eigenproblem (4) have trivial intersection. The proof is the same as for the classical GenEO coarse space (see [3, Lem. 3.18]).

Lemma 3 *For $j \in \{1, \dots, N\}$ let $b(\cdot, \cdot) = a_{\Omega_j^*}(\cdot, \cdot)$ and $b(\cdot, \cdot) = a_{\Omega_j^*}(\eta_j(\cdot), \eta_j(\cdot))$. Then* $\ker b \cap \ker c = \{0\}$.

Proof. For subdomains where the extended overlapping zone Ω_j^* touches the global Dirichlet boundary, $b(\cdot, \cdot)$ is positive definite, hence $\ker b = \{0\}$. Otherwise, $\ker b = \operatorname{span}\{\mathbb{1}\}$, where $\mathbb{1}$ is the constant one function on Ω_j. But $\mathbb{1} \notin \ker c$ due to the modified partition of unity function η_j. □

Thus we can apply Lemma 2 to the eigenproblem (4).

Corollary 1 *Let* $m_j \in \{1, \dots, \dim(V_h(\Omega_j^*))\}$ *be such that* $0 < \lambda^j_{m_j+1} < \infty$. *The local projection operator* $\Pi^*_{j,m_j} : V_h(\Omega_j^*) \to V_h(\Omega_j^*)$,

$$\Pi^*_{j,m_j} v \coloneqq \sum_{k=1}^{m_j} a_{\Omega_j^*}(\eta_j(v), \eta_j(t_k^j)) t_k^j \,,$$

is well-defined and orthogonal w.r.t. the bilinear form $a_{\Omega_j^*}(\cdot, \cdot)$. *Thus*

$$|\Pi^*_{j,m_j} v|_{a,\Omega_j^*} \le |v|_{a,\Omega_j^*} \quad \textit{and} \quad |v - \Pi^*_{j,m_j} v|_{a,\Omega_j^*} \le |v|_{a,\Omega_j^*}$$

and we have the local stability estimate

$$|\eta_j(v - \Pi^*_{j,m_j} v)|^2_{a,\Omega_j^*} \le 1/\lambda^j_{m_j+1} |v - \Pi^*_{j,m_j} v|^2_{a,\Omega_j^*} \,.$$

The operator Π^*_{j,m_j} only maps to $V_h(\Omega_j^*)$ and as such only defines the coarse components in Ω_j^*. We thus additionally define

$$\Pi_{j,m_j} v \coloneqq \sum_{k=1}^{m_j} a_{\Omega_j^*}(\eta_j(v), \eta_j(t_k^j)) y_k^j \,.$$

We can now define a stable decomposition.

Theorem 2 *Let* $v \in V_h(\Omega)$. *The decomposition*

$$z_0 \coloneqq \sum_{j=1}^{N} \xi_j(\Pi_{j,m_j} v|_{\Omega_j}) \,, \qquad z_j \coloneqq \xi_j(v|_{\Omega_j} - \Pi_{j,m_j} v|_{\Omega_j}), \quad \textit{for } j = 1, \dots, N \,,$$

is C_0*-stable with*

$$C_0^2 = 2 + k_0(2k_0 + 1) \max_{1 \le j \le N} (2 + 3/\lambda^j_{m_j+1}) \,.$$

Proof. Since ξ_j is the identity for restrictions of functions to ω_j° we have

$$|z_j|^2_{a,\Omega_j} = \left|\xi_j(v - \Pi_{j,m_j} v)\right|^2_{a,\Omega_j^\circ} + \left|v - \Pi_{j,m_j} v\right|^2_{a,\omega_j^\circ} \,. \tag{6}$$

We will treat each term separately. For the first part, we use that $\xi_j = \eta_j$ and $t_k^j = y_k^j$ on Ω_j° so that by Lemma 2 we have

$$\begin{aligned} \left|\xi_j(v - \Pi_{j,m_j} v)\right|^2_{a,\Omega_j^\circ} &= \left|\eta_j(v - \Pi^*_{j,m_j} v)\right|^2_{a,\Omega_j^\circ} \le \left|\eta_j(v - \Pi^*_{j,m_j} v)\right|^2_{a,\Omega_j^*} \\ &\le 1/\lambda^j_{m_j+1} |v|^2_{a,\Omega_j^*} \le 1/\lambda^j_{m_j+1} |v|^2_{a,\Omega_j} \,. \end{aligned}$$

For the second part, we estimate

$$\begin{aligned}
\left|v - \Pi_{j,m_j} v\right|^2_{a,\omega_j^\circ} &= \left|v - \Pi_{j,m_j} v + H_j^\circ v - H_j^\circ v\right|^2_{a,\omega_j^\circ} \\
&= \left|(I - H_j^\circ) v + H_j^\circ (v - \Pi_{j,m_j} v)\right|^2_{a,\omega_j^\circ} \\
&= \left|(I - H_j^\circ) v + H_j^\circ E_j^\circ \gamma_j^\circ (v - \Pi_{j,m_j} v)\right|^2_{a,\omega_j^\circ} \\
&= \left|(I - H_j^\circ) v + H_j^\circ E_j^\circ (\gamma_j^\circ v - \gamma_j^\circ \Pi^*_{j,m_j} v)\right|^2_{a,\omega_j^\circ} \\
&\leq 2\left|(I - H_j^\circ) v\right|^2_{a,\omega_j^\circ} + 2\left|H_j^\circ E_j^\circ (\gamma_j^\circ v - \gamma_j^\circ \Pi^*_{j,m_j} v)\right|^2_{a,\omega_j^\circ} \\
&\leq 2|v|^2_{a,\omega_j^\circ} + 2\left|\eta_j (v - \Pi^*_{j,m_j} v)\right|^2_{a,\omega_j^\circ \cap \Omega_j^*} \\
&\leq 2|v|^2_{a,\Omega_j} + 2\left|\eta_j (v - \Pi^*_{j,m_j} v)\right|^2_{a,\Omega_j^*} \leq (2 + 2/\lambda^j_{m_j+1}) |v|^2_{a,\Omega_j} ,
\end{aligned}$$

where we used that $I - H_j^\circ$ is an a-orthogonal projection on ω_j°, $\Pi_{j,m_j} v = H_j^\circ \Pi_{j,m_j} v$, $H_j^\circ v = H_j^\circ E_j^\circ \gamma_j^\circ v$, $\gamma_j^\circ \Pi_{j,m_j} v = \gamma_j^\circ \Pi^*_{j,m_j} v$, and the energy-minimality of the harmonic extension. Then we extend the domains in the seminorms and apply Lemma 2. Plugging everything into (6) yields the result. □

Corollary 2 *The condition number of the two-level Schwarz method with the R-GenEO coarse space can be bounded as*

$$\kappa(\mathbf{M}^{-1}_{\mathrm{AS},2}\mathbf{A}) \leq (1 + k_0)(2 + k_0(2k_0 + 1) \max_{1 \leq j \leq N} (2 + 3/\lambda^j_{m_j+1})) .$$

3 Numerical results

We consider a diffusion problem with variable high-contrast diffusion coefficient, see [3, Section 5.1] for a detailed description. We discretise the equation using $\mathbb{Q}_1$ finite elements on a structured grid.

Table 1 reports the number of iterations to solve the resulting linear system using the conjugate gradient (CG) method preconditioned with a two-level Schwarz method using both the classical GenEO coarse space and the R-GenEO coarse space. Both methods are implemented in an MPI-parallel code based on the Dune framework [2]. The eigenvalue problems are solved using the Spectra library (https://spectralib.org) by rewriting them as standard eigenvalue problems using a shift-invert spectral transformation. UMFPACK [5] is used to factorise and solve the shift-invert operators (because the operator is not necessarily positive definite), and CHOLMOD [4] is used for the subdomain problems and the coarse problem. The stopping criterion is a relative residual reduction of 10^{-8}. We partition the domain into a regular grid of 2 × 2, 4 × 4, 8 × 8 or 16 × 16 non-overlapping subdomains and then add two layers of elements to create the overlapping subdomains. The number of elements per subdomain is 256×256 in all tests. We use a fixed number of eigenfunctions per subdomain to set up the coarse spaces.

	24 eigenvectors						8 eigenvectors					
	GenEO			R-GenEO			GenEO			R-GenEO		
N	# its.	t_s	t_ℓ	# its.	t_s	t_ℓ	# its.	t_s	t_ℓ	# its.	t_s	t_ℓ
4	18	2.03	0.19	19	0.92	0.18	26	1.29	0.22	23	0.66	0.20
16	27	2.47	0.35	28	1.26	0.36	41	1.43	0.45	44	0.72	0.51
64	29	2.93	0.50	30	1.50	0.52	47	1.63	0.74	50	0.83	0.75
256	32	2.95	0.63	31	1.67	0.61	60	1.66	0.98	57	0.86	0.95

Table 1 Comparison of GenEO and R-GenEO with 24 and 8 eigenvectors per subdomain. N: number of subdomains, # its.: number of CG iterations, t_s: setup time [s], t_ℓ: solve time [s].

It can be observed that the variant based on the R-GenEO coarse space converges in a similar number of iterations while the setup-time is reduced by a factor 2, independent of the number of eigenvectors per subdomain. Note that in this example the R-GenEO-based method sometimes requires less iterations than the variant based on the classical GenEO coarse space. As mentioned above, for certain examples (see [6] and [1]) the method is expected to require more iterations and/or a larger coarse space since the eigenproblem might not “see” certain parts of the coefficient.

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