

Analysis of Two-Level Schwarz Preconditioners for Space-Time Discontinuous Galerkin Method

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1 Introduction

The space-time discontinuous Galerkin (STDG) method exhibits an efficient technique for the numerical solution of time-dependent partial differential equations due to its flexibility, stability, high order of accuracy with respect to space and time, and a natural ability to treat varying nonnested meshes at different time levels. On the other hand, STDG discretization leads to larger algebraic systems in comparison to, e.g., diagonally implicit Runge-Kutta schemes. Therefore, for the STDG method to be competitive, it is necessary to use effective preconditioners for algebraic solvers.

Very popular are two-level Schwarz preconditioners which allow a parallelization in a simple way, namely, in the context of the discontinuous Galerkin (DG) method. Nevertheless, the time approximation of degree $q \geq 1$ leads to non-symmetric algebraic systems even for symmetric spatial operators, which does not allow to employ the standard tools of numerical analysis, cf. [10, Chapter 11] and also [1, 2].

In this contribution, we consider the parabolic initial boundary value problem that models heat conduction in a spatial domain $\Omega \subset \mathbb{R}^d$ over a time interval $(0, T)$. The problem is to find a function $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} \partial_t u - \epsilon \Delta u &= f && \text{in } Q_T := \Omega \times (0, T), \\ u(x, 0) &= u^0(x) && \text{in } \Omega, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, T), \end{aligned} \tag{1}$$

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where $f : \Omega \times (0, T) \rightarrow \mathbb{R}$ is the given function and $u_0 : \Omega \rightarrow \mathbb{R}$ is the given initial condition.

We discretize (1) using the STDG method and study the convergence of the arising algebraic systems that are preconditioned by two-level additive and hybrid Schwarz preconditioners. The hybrid preconditioner is additive with respect to the local components and multiplicative with respect to the mesh levels. In Section 4, we recall the standard results relating to the convergence of the GMRES method from [10] and show that the assumptions of this theorem are valid only for sufficiently fine coarse grid. However, although the assumptions are not valid, the numerical experiments in Section 5 show that the preconditioner works quite well.

2 STDG discretization

We briefly present the space-time discontinuous Galerkin discretization of (1), and refer to [4, Chapter 6] for details. Let $r > 0$ be an integer and consider the partition $0 = t_0 < t_1 < \dots < t_r = T$, time intervals $I_m = (t_{m-1}, t_m)$, $\tau = t_m - t_{m-1}$, and $Q_m = \Omega \times I_m$ for $m = 1, \dots, r$. For each time level $m = 0, \dots, r$, we consider the triangular mesh $\mathcal{T}_{h,m} = \{\mathcal{K}\}$ of the spatial domain Ω , and define a space of piecewise polynomial functions

$$S_{h,m} = \{\varphi \in L^2(\Omega); \varphi|_{\mathcal{K}} \in P_p(\mathcal{K}) \forall \mathcal{K} \in \mathcal{T}_{h,m}\}, \quad m = 1, \dots, r. \quad (2)$$

Moreover, we consider the spaces of space-time piece-wise polynomial discontinuous functions (spatial degree p and temporal degree q) on Q_m

$$S_{h\tau,m} = \left\{ \varphi \in L^2(Q_m); \varphi(x, t) = \sum_{j=0}^q t^j \varphi_j(x), \varphi_j \in S_{h,m}, i = 0, \dots, q \right\}, \quad (3)$$

for $m = 1, \dots, r$. We denote the traces of $\varphi \in S_{h\tau,m}$ with respect to time by $\varphi|_{m}^- = \lim_{t \rightarrow t_m^-} \varphi(t)$, and $\varphi|_{m-1}^+ = \lim_{t \rightarrow t_{m-1}^+} \varphi(t)$. For each $m = 1, \dots, r$, we define the following bilinear form

$$\mathcal{B}_m^\tau(u, \varphi) := \int_{I_m} ((u', \varphi) + \mathcal{A}_{h,m}(u, \varphi)) \, dt + (u|_{m-1}^+, \varphi|_{m-1}^+), \quad (4)$$

where (\cdot, \cdot) is the $L^2(\Omega)$ -scalar product and $\mathcal{A}_{h,m}(u, \varphi)$ is the symmetric bilinear form arising from the discretization of the diffusive term in (1), cf. [4, Chapter 2]. Then the discrete problem reads as follows.

Definition 1. Find $u = \{u_m\}_{m=1}^r$, $u_m \in S_{h\tau,m}$, $m = 1, \dots, r$ such that

$$\mathcal{B}_m^\tau(u, \varphi) = \int_{I_m} (f, \varphi) \, dt + (u|_{m-1}^-, \varphi|_{m-1}^+) \quad \forall \varphi \in S_{h\tau,m}, \quad m = 1, \dots, r, \quad (5)$$

where $u|_0^- = u^0$ is given by the initial condition.

We note that for nonhomogeneous Dirichlet and/or Neumann boundary conditions, the right-hand side of (5) contains additional terms arising from the boundary conditions, cf. [4, Chapter 2]. Moreover, we introduce the symmetric part of \mathcal{B}_m^τ as

$$\mathcal{A}_m^\tau(u, \varphi) = \int_{I_m} \mathcal{A}_{h,m}(u, \varphi) dt + (u|_{m-1}^+, \varphi|_{m-1}^+). \quad (6)$$

The algebraic representations of the forms \mathcal{B}_m^τ and \mathcal{A}_m^τ in a suitable basis of $S_{h\tau,m}$ are denoted by \mathbf{B}_m and \mathbf{A}_m , respectively. Then, we write problem (5) as the system of linear algebraic equations $\mathbf{B}_m \mathbf{u} = \mathbf{f}$, $m = 1, \dots, r$, where \mathbf{u} and \mathbf{f} are algebraic representations of solution $u \in S_{h\tau,m}$ and the right-hand side of (5), respectively. Since we consider non-adapted meshes $\mathcal{T}_{h,m}$, we omit the subscript m and write \mathbf{B} and \mathbf{A} instead of \mathbf{B}_m and \mathbf{A}_m , respectively, for all $m = 1, \dots, r$.

3 Domain decomposition

For each time level $m = 1, \dots, r$, we solve the linear algebraic system $\mathbf{B}\mathbf{u} = \mathbf{f}$ iteratively using the GMRES method with a domain decomposition preconditioner. We consider only the spatial domain decomposition for each time slab I_m separately.

We directly extend the approach, for example, from [3], to the STDG method. We consider non-overlapping subdomains Ω_i , $i = 1, \dots, N$ that cover the computational domain Ω . The subdomains Ω_i , $i = 1, \dots, N$ are constructed by connecting elements of the mesh \mathcal{T}_h , we use METIS library [6]. We denote by $Q_m^i = \Omega_i \times I_m$ the space-time subdomain cylinder.

Furthermore, we define the local spaces for each subdomain Ω_i by (cf. (2) – (3))

$$S_{h,m}^i = \{ \varphi \in L^2(\Omega_i); \varphi|_{\mathcal{K}} \in P_p(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_{h,m} \cap \Omega_i \}, \quad i = 1, \dots, N, \quad (7)$$

and

$$S_{h\tau,m}^i = \left\{ \varphi \in L^2(Q_m^i); \varphi(x, t) = \sum_{j=0}^q t^j \varphi_j(x), \varphi_j \in S_{h,m}^i \right\}, \quad i = 1, \dots, N, \quad (8)$$

for $m = 1, \dots, r$. Additionally, we consider a coarse mesh \mathcal{T}_H of Ω where $H = \max_{\mathcal{K} \in \mathcal{T}_H} \text{diam}(\mathcal{K})$ denotes the coarse mesh step size. The coarse mesh is constructed such that each coarse element is a union of fine elements of $\mathcal{T}_h \cap \Omega_i$ for some $i = 1, \dots, N$. Therefore, $\mathcal{K} \in \mathcal{T}_H$ are typically polygonal elements, possibly non-convex. Then the global coarse analogs of spaces (7) and (8) are $S_{h,m}^0$ and $S_{h\tau,m}^0$, where we use Q_m instead of Q_m^i and $\mathcal{T}_{H,m}$ instead of $\mathcal{T}_{h,m}$. Obviously, due to DG discretization, we have $S_{h,m}^0 \subset S_{h,m}$ and $S_{h\tau,m}^0 \subset S_{h\tau,m}$, which significantly simplifies the construction of the global coarse problems.

Let $m = 1, \dots, r$ be arbitrary but fixed. In the following, we do not express the dependence of the symbols introduced hereafter on m . We define the restriction operators $R_i : S_{h\tau,m} \rightarrow S_{h\tau,m}^i$, $i = 1, \dots, N$ as the operator that restricts the function from the global space to the local space on the subdomain Ω_i . The prolongation

operators $R_i^T : S_{h\tau,m}^i \rightarrow S_{h\tau,m}$, $i = 1, \dots, N$ are defined as an extension of the function $v_i \in S_{h,i}$ by zero on $\Omega \setminus \Omega_i$. Moreover, the coarse prolongation operator $R_0^T : S_{h\tau,m}^0 \rightarrow S_{h\tau,m}$ is defined as the injection from the coarse space to the fine space. The coarse restriction operator $R_0 : S_{h\tau,m} \rightarrow S_{h\tau,m}^0$ is the adjoint operator to R_0^T . The algebraic representations of the operators R_i and R_i^T in the basis of $S_{h\tau,m}$ are denoted by the matrices \mathbf{R}_i and \mathbf{R}_i^T , $i = 0, \dots, N$, respectively.

In virtue of (4), we define the local bilinear forms by

$$\mathcal{B}_{m,i}^\tau(u_i, v_i) = \mathcal{B}_m^\tau(\mathbf{R}_i^T u_i, \mathbf{R}_i^T v_i), \quad u_i, v_i \in S_{h\tau,m}^i, \quad i = 0, \dots, N, \quad (9)$$

their algebraic representation are given by $\mathbf{B}_i = \mathbf{R}_i \mathbf{B} \mathbf{R}_i^T$, $i = 0, \dots, N$.

Furthermore, we define the local projection operators $\tilde{P}_i : S_{h\tau,m} \rightarrow S_{h\tau,m}^i$ as

$$\mathcal{B}_{m,i}^\tau(\tilde{P}_i u, v_i) = \mathcal{B}_m^\tau(u, \mathbf{R}_i^T v_i) \quad \forall v_i \in S_{h\tau,m}^i, \quad i = 0, \dots, N, \quad (10)$$

and the global projection operator $P_i : S_{h\tau,m} \rightarrow S_{h\tau,m}$ as, $P_i = \mathbf{R}_i^T \tilde{P}_i$, $i = 0, \dots, N$. The matrix representation of the operators \tilde{P}_i and P_i is given by $\tilde{\mathbf{P}}_i = \mathbf{B}_i^{-1} \mathbf{R}_i \mathbf{B}$ and $\mathbf{P}_i = \mathbf{R}_i^T \mathbf{B}_i^{-1} \mathbf{R}_i \mathbf{B}$, $i = 0, \dots, N$, respectively.

According to [10, Chapter 2], the two-level additive Schwarz operator and the two-level symmetric hybrid Schwarz operator are given by

$$P_{\text{add}} = \sum_{i=0}^N P_i, \quad \text{and} \quad P_{\text{hyb}} = P_0 + (I - P_0) \sum_{i=1}^N P_i (I - P_0), \quad (11)$$

respectively. From the matrix representation of P_i we find that the matrix representation of the additive and hybrid Schwarz operator can be written as

$$\mathbf{P}_* \mathbf{u} = \mathbf{N}_*^{-1} \mathbf{B} \mathbf{u} = \mathbf{N}_*^{-1} \mathbf{f}, \quad (12)$$

where we use the notation \mathbf{P}_* for the additive or hybrid Schwarz operator and \mathbf{N}_*^{-1} for the respective preconditioners. It should be noted that the matrix \mathbf{B} is not symmetric, and therefore we use GMRES to solve the preconditioned system (12).

4 Generalized minimal residual method (GMRES)

We consider the solution linear system (12) by the GMRES method, cf. [7, Chapter 6.5]. Following [5, Theorem 3.3], the convergence of the GMRES method can be described by the minimal eigenvalue of the symmetric part of the system matrix and the norm of the system matrix, i. e.

$$\|\mathbf{r}_k\|_2 \leq \left(1 - \frac{\lambda_{\min}(\mathbf{H})^2}{\lambda_{\max}(\mathbf{B}^T \mathbf{B})} \right)^{k/2} \|\mathbf{r}_0\|_2 \quad (13)$$

where $\mathbf{H} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$ and $\mathbf{r}_k = \mathbf{b} - \mathbf{B}\mathbf{x}_k$. However, we are interested in the bound on the preconditioned system. Hence, we define these quantities in a continuous setting and then translate them to the algebraic setting. For the purpose of the analysis, we consider the bilinear form $\mathcal{A}_m^\tau(\cdot, \cdot)$ defined in (6) as an inner product used in the GMRES algorithm.

We start with the Elman convergence bound for the GMRES method. Let $\mathbf{r}_k = \mathbf{N}_*^{-1}(\mathbf{f} - \mathbf{B}\mathbf{u}_k)$ be the k -th preconditioned residual, $k = 0, 1, \dots$ of the preconditioned GMRES method. As we mentioned above, we state the convergence bound in the continuous setting; see [10, Theorem 11.1, C.11].

Theorem 1 (GMRES convergence bound) *There exist constants $H_0 > 0$, $\underline{c}(H_0) > 0$, and $\bar{c}(H_0) > 0$, such that*

$$\underline{c}(H_0)\mathcal{A}_m^\tau(u, u) \leq \mathcal{A}_m^\tau(P_*u, u) \quad \text{and} \quad \mathcal{A}_m^\tau(P_*u, P_*u) \leq \bar{c}(H_0)\mathcal{A}_m^\tau(u, u) \quad (14)$$

for any coarse mesh step size $H \leq H_0$ and any $u \in S_{h\tau, m}$. Then the k -th preconditioned residual of the GMRES algorithm fulfills the bound

$$\|\mathbf{r}_k\|_A \leq \left(1 - \underline{c}(H_0)^2/\bar{c}(H_0)\right)^{k/2} \|\mathbf{r}_0\|_A, \quad (15)$$

where $\|\cdot\|_A$ is the norm induced by the bilinear form $\mathcal{A}_m^\tau(\cdot, \cdot)$.

Taking into account the coefficients in (14), the upper bound $\bar{c}(H_0)$ can be easily obtained and has been analytically shown in [1, Section 6.3]. Moreover, the constant is upper bound on the operator norm and is analogous to $\lambda_{\max}(\mathbf{B}^T\mathbf{B})$ in (13) for not-preconditioned system. However, the lower bound $\underline{c}(H_0)$ cannot be guaranteed analytically.

Moving to the algebraic setting, we can express the constant $\underline{c}(H_0)$ as

$$\underline{c}(H_0) \leq \frac{\mathcal{A}_m^\tau(P_*u, u)}{\mathcal{A}_m^\tau(u, u)} = \frac{\mathbf{u}^T(\mathbf{A}P_*)\mathbf{u}}{\mathbf{u}^T\mathbf{A}\mathbf{u}} = \frac{\mathbf{u}^T(\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B})\mathbf{u}}{\mathbf{u}^T\mathbf{A}\mathbf{u}} \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (16)$$

Moreover, for real matrices, we have the following relation

$$\frac{\mathbf{u}^T(\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B})\mathbf{u}}{\mathbf{u}^T\mathbf{A}\mathbf{u}} = \frac{\frac{1}{2}\mathbf{u}^T(\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B} + (\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B})^T)\mathbf{u}}{\mathbf{u}^T\mathbf{A}\mathbf{u}} \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (17)$$

Using the fact that $(\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B} + (\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B})^T)$ is a symmetric matrix we can bound from below (17) by its minimal generalized eigenvalue, which is analogue to the not-preconditioned case $\lambda_{\min}(\mathbf{H})$ in (13). Hence we consider the generalized eigenvalue problem: Find $(\mathbf{u}, \lambda) \in \mathbb{R}^n \times \mathbb{R}$, $(\mathbf{u}, \lambda) \neq (\mathbf{0}, 0)$ such that

$$\frac{1}{2}(\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B} + (\mathbf{A}\mathbf{N}_*^{-1}\mathbf{B})^T)\mathbf{u} = \lambda\mathbf{A}\mathbf{u}, \quad (18)$$

where \mathbf{u} is the generalized eigenvector and λ is the generalized eigenvalue. Obviously, relations (16) – (18) imply the inequality $\underline{c}(H_0) \leq \lambda$ for any generalized eigenvalue

λ of problem (18). Therefore, if the minimal generalized eigenvalue of (18) λ_{min} is negative, estimate (16) is violated, and the first assumption in (14) is not valid. For more information on the bound of GMRES computation using the symmetric part of the system matrix and the lowest generalized eigenvalue see [8].

We demonstrate numerically that the lower bound can be below zero for some parameter settings.

5 Numerical results

In this section, we present numerical experiments related to the dependence of the estimate of the lower bound $\underline{c}(H_0)$ in (14) (given by λ_{min} = minimal generalized eigenvalue of (18)) on the ratio between the coarse and fine mesh steps H/h and other discretization parameters. In particular, we show that the size of the coarse mesh significantly influences the lower bound, leading to nearly impractical coarse mesh restrictions given by Theorem 1. We compute the generalized eigenvalue problem (18) using the Matlab eigenvalue solver for sparse matrices (function `eigs`). The solver is based on the Krylov-Schur algorithm, see [9], and is suitable for large-scale sparse problems.

We consider the model problem (1) with non-homogeneous Dirichlet boundary conditions on the unit square $\Omega = (0, 1)^2$ with $T = 1$. The boundary conditions and the right-hand side are chosen such that the exact solution is $u(x_1, x_2, t) = \exp(2t + x_1 + x_2)$. We perform calculations using a sequence of three uniform meshes that have rectangular triangles $\#\mathcal{T}_h = 126, 256,$ and 512 . We set $\epsilon = 1$, $N = 10$, and subdivide each subdomain into 1, 2, 4, and 8 coarse elements. We use a piecewise quadratic approximation with respect to space and time, i.e., $p = q = 2$. We investigate the dependence of the estimate of the lower bound $\underline{c}(H_0)$ (cf. (14)) on $H/h \approx \sqrt{\#\mathcal{T}_h/\#\mathcal{T}_H}$, and also on the time step size chosen as $\tau = 0.5, 0.05, 0.005$.

		Additive Schwarz				Hybrid Schwarz			
τ	$\#\mathcal{T}_h$	$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 20$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$	$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 20$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$
0.5	126	0.021	0.037	0.063	0.063	0.026	0.045	0.074	0.071
	256	0.015	0.020	0.035	0.054	0.017	0.022	0.038	0.063
	512	0.011	0.019	0.028	0.044	0.012	0.021	0.032	0.054
0.05	126	-0.006	0.029	0.062	0.060	0.013	0.044	0.073	0.070
	256	-0.022	-0.001	0.031	0.054	-0.003	0.013	0.036	0.063
	512	-0.022	0.013	0.027	0.043	-0.008	0.018	0.032	0.053
0.005	126	-0.702	-0.656	-0.513	-0.385	-0.196	-0.044	0.057	0.043
	256	-0.824	-0.753	-0.634	-0.489	-0.267	-0.172	-0.017	0.048
	512	-0.842	-0.757	-0.691	-0.543	-0.257	-0.085	-0.013	0.041

Table 1 Comparison of the numerical computation of constant $\underline{c}(H_0)$ for different meshes, preconditioners and time steps.

The results presented in Table 1 show that a larger value of $\#\mathcal{T}_H$ increases the estimate of $\underline{c}(H_0)$, which is in agreement with the expectation, since the coarser mesh provides more information. On the other hand, the decrease in the time step causes a decrease in the estimate of $\underline{c}(H_0)$, and it becomes even negative, which does not admit to use the convergence bound of Theorem 1.

ϵ	$\#\mathcal{T}_h$	Additive Schwarz				Hybrid Schwarz			
		$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 20$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$	$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 20$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$
1	126	0.021	0.037	0.063	0.063	0.026	0.045	0.074	0.071
	256	0.015	0.020	0.035	0.054	0.017	0.022	0.038	0.063
	512	0.011	0.019	0.028	0.044	0.012	0.021	0.032	0.054
0.1	126	-0.601	-0.562	-0.559	-0.559	-0.196	-0.044	0.057	0.043
	256	-0.661	-0.603	-0.566	-0.564	-0.267	-0.172	-0.017	0.048
	512	-0.672	-0.564	-0.563	-0.565	-0.257	-0.085	-0.013	0.041

Table 2 Comparison of the numerical computation of constant $\underline{c}(H_0)$ for different meshes, preconditioners and values of diffusion coefficients.

In the next series of experiments, we fix $\tau = 0.05$ and investigate the dependence of the estimate of $\underline{c}(H_0)$ on the diffusion coefficient ϵ . We compare two different values of the coefficient $\epsilon = 1$ and 0.1 as it is enough to demonstrate the dependence on the coarse mesh size and the limitations of this approach. The results are presented in Table 2. The lower diffusion coefficient gives a negative lower bound $\underline{c}(H_0)$ for additive Schwarz preconditioner for all coarse mesh sizes tested.

Although the experimental values of $\underline{c}(H_0)$ indicate that the first inequality in (14) in Theorem 1 is not valid for STDG discretization in general, the next series of numerical experiments shows that both preconditioners converge efficiently. In particular, we investigate the number of iterations of preconditioned GMRES necessary to achieve $\|\mathbf{r}_k\|/\|\mathbf{r}_0\| < 10^{-6}$.

Table 3 shows the number of GMRES iterations for different time steps and $\#\mathcal{T}_H$ while the other parameters are fixed ($N = 10$, $\#\mathcal{T}_h = 512$, $p = q = 2$, $\epsilon = 1$). We observe that increasing the number of $\#\mathcal{T}_H$ decreases the number of GMRES iterations due to the more information provided by the coarse grid. On the other hand, the size of the time step has only a mild influence on the speed of convergence, which is an interesting observation, and additional research is needed to investigate this behavior.

τ	Additive Schwarz			Hybrid Schwarz			Not-preconditioned
	$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$	$\#\mathcal{T}_H = 10$	$\#\mathcal{T}_H = 40$	$\#\mathcal{T}_H = 80$	
0.5	67	44	36	25	12	8	3984
0.05	71	46	36	27	13	8	1877
0.005	65	47	37	27	13	8	517
0.0005	41	37	33	19	11	7	148

Table 3 Number of GMRES iterations on one time level.

6 Conclusion

We presented a numerical study of additive and hybrid Schwarz preconditioners for the space-time DG method showing the limitations of the theoretical approach based on the minimal eigenvalue of the generalized eigenvalue problem. However, additional numerical experiments show that preconditioned GMRES converges in all the tested cases. The hybrid Schwarz preconditioner seems to be more robust than the additive one. Additional research is needed to provide theoretical bounds for the convergence of the GMRES method preconditioned by the Schwarz preconditioners.

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References

1. Antonietti, P.F., Ayuso, B.: Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: Non-overlapping case. *ESAIM: M2AN* **41**(1), 21–54 (2007)
2. Antonietti, P.F., Ayuso, B.: Multiplicative schwarz methods for discontinuous Galerkin approximations of elliptic problems. *ESAIM: M2AN* **42**(3), 443–469 (2008)
3. Antonietti, P.F., Houston, P.: A class of domain decomposition preconditioners for hp -discontinuous Galerkin finite element methods. *J. Sci. Comput.* **46**(1), 124–149 (2011)
4. Dolejší, V., Feistauer, M.: Discontinuous Galerkin method. Analysis and applications to compressible flow. Berlin: Springer (2015)
5. Eisenstat, S.C., Elman, H.C., Schultz, M.H.: Variational iterative methods for nonsymmetric systems of linear equations. *SIAM J. Numer. Anal.* **20**(2), 345–357 (1983)
6. Karypis, G., Kumar, V.: METIS – A Software Package for Partitioning Unstructured Graphs, Partitioning Meshes, and Computing Fill-Reducing Orderings of Sparse Matrices (2011). <<http://glaros.dtc.umn.edu/gkhome/metis/metis/overview>>
7. Saad, Y.: Iterative methods for sparse linear systems, second edn. Society for Industrial and Applied Mathematics, Philadelphia, PA (2003)
8. Starke, G.: Field-of-values analysis of preconditioned iterative methods for nonsymmetric elliptic problems. *Num. Math.* **78**(1), 103–117 (1997)
9. Stewart, G.W.: A Krylov–Schur algorithm for large eigenproblems. *SIAM J. Matrix Anal. Appl.* **23**(3), 601–614 (2002)
10. Toselli, A., Widlund, O.: Domain decomposition methods—algorithms and theory, *Springer Series in Computational Mathematics*, vol. 34. Springer-Verlag, Berlin (2005)