Numerical Solution of Elliptic Distributed Optimal Control Problems with Boundary Value Tracking

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1 Introduction

Let us consider the following boundary tracking optimal control problem (ocp): Find the state $y_\varrho \in H^1(\Omega)$ and the optimal control $u_\varrho \in U$ minimizing the cost functional

$$J(y_\varrho, u_\varrho) = \frac{1}{2}\, \|y_\varrho - \overline{y}\|^2_{L^2(\Gamma)} + \frac{1}{2}\, \varrho\, \|u_\varrho\|^2_U \tag{1}$$

subject to the Neumann boundary value problem (bvp)

$$-\Delta y_\varrho + y_\varrho = u_\varrho \quad \text{in } \Omega, \quad \partial_n y_\varrho = 0 \quad \text{on } \Gamma, \tag{2}$$

where $\overline{y} \in L^2(\Gamma)$ denotes a given target, $\varrho \in \mathbb{R}_+$ is a positive regularization or cost parameter, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded Lipschitz domain with the boundary $\Gamma = \partial\Omega$. Our work was inspired by the paper [4], where the L^2 regularization corresponding to the choice $U = L^2(\Omega)$ was investigated. The boundary tracking is a special case of the partial tracking of a given target in a subset of Ω that is also called limited observation; see, e.g., [5]. There are many important practical applications, e.g., inverse heat transfer, photoacoustic tomography, etc., see also [4]. In this contribution, we consider the energy regularization corresponding to the choice $U = \widetilde{H}^{-1}(\Omega) := [H^1(\Omega)]^*$ such that the solution of (2) defines an isomorphism when considering $y_\varrho \in H^1(\Omega)$. When using $U = L^2(\Omega)$ this implies

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$y_\varrho \in H^1_\Delta(\Omega) := \{y \in H^1(\Omega) : \Delta y \in L^2(\Omega)\}$ to ensure an isomorphism, which afterwards requires higher-order basis functions for a conforming discretization. Here we use the standard notations for Lebesgue and Sobolev spaces; see, e.g., [8]. In order to follow the abstract theory presented in [3], we define the state space in such a way that the state-to-control map is an isomorphism. This allows us to derive a state-based formulation which is the basis for the numerical solution. Here we restrict the analysis to a conforming tensor-product finite element (fe) discretization that finally leads to a linear system of algebraic equations for which fast solvers can be constructed. Note that c denotes a universal constant independent of the regularization parameter ϱ, and the discretization parameter h.

2 State-based variational reformulation

The variational formulation of the Neumann boundary value problem (2) reads to find $y_\varrho \in H^1(\Omega)$ such that

$$\langle \nabla y_\varrho, \nabla y\rangle_{L^2(\Omega)} + \langle y_\varrho, y\rangle_{L^2(\Omega)} = \langle u_\varrho, y\rangle_\Omega \tag{3}$$

is satisfied for all $y \in H^1(\Omega)$, where we assume $u_\varrho \in \widetilde{H}^{-1}(\Omega)$. The variational problem has a unique solution due to Lax-Milgram's lemma; see, e.g., [8]. While the Neumann boundary condition in (2) enters the variational formulation (3) in a natural way, this condition has to be included in the definition of the state space

$$Y := \Big\{ y \in H^1(\Omega) : \langle \partial_n y, \phi\rangle_\Gamma = 0 \text{ for all } \phi \in H^{1/2}(\Gamma) \Big\}.$$

When using duality arguments, we then conclude

$$\|u_\varrho\|_{Y^*} := \sup_{0 \neq y \in Y} \frac{\langle u_\varrho, y\rangle_\Omega}{\|y\|_{H^1(\Omega)}} = \sup_{0 \neq y \in Y} \frac{\langle y_\varrho, y\rangle_{H^1(\Omega)}}{\|y\|_{H^1(\Omega)}} = \|y_\varrho\|_{H^1(\Omega)},$$

and instead of (1) we can consider the reduced state-based cost functional

$$\widetilde{J}(y_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(\Gamma)} + \frac{1}{2}\,\varrho\,\|y_\varrho\|^2_{H^1(\Omega)}, \tag{4}$$

whose minimizer $y_\varrho \in Y$ is the unique solution of the gradient equation satisfying

$$\langle y_\varrho, y\rangle_{L^2(\Gamma)} + \varrho\,\langle y_\varrho, y\rangle_{H^1(\Omega)} = \langle \overline{y}, y\rangle_{L^2(\Gamma)} \quad \text{for all } y \in Y. \tag{5}$$

Following the abstract theory as given in [3, Lemma 2.1], we have the following regularization error estimate.

Lemma 1 *Let $y_\varrho \in Y$ be the unique solution of the variational formulation* (5)*. For $\overline{y} \in L^2(\Gamma)$, there hold the estimates*

$$\|y_\varrho - \overline{y}\|_{L^2(\Gamma)} \leq \|\overline{y}\|_{L^2(\Gamma)}, \quad \|y_\varrho\|_{H^1(\Omega)} \leq \varrho^{-1/2}\,\|\overline{y}\|_{L^2(\Gamma)}. \tag{6}$$

If $\overline{y} \in H^{1/2}(\Gamma)$ *is the Dirichlet trace of* $\overline{y}_e \in Y$*, then* $\|y_\varrho\|_{H^1(\Omega)} \leq \|\overline{y}_e\|_{H^1(\Omega)}$*, and*

$$\|y_\varrho - \overline{y}\|_{L^2(\Gamma)} \leq \varrho^{1/2}\,\|\overline{y}_e\|_{H^1(\Omega)}, \quad \|y_\varrho - \overline{y}_e\|_{H^1(\Omega)} \leq \|\overline{y}_e\|_{H^1(\Omega)}. \tag{7}$$

If $\overline{y} \in H^1(\Gamma)$ *is the Dirichlet trace of* $\overline{y}_e \in Y \cap H^{3/2+\varepsilon}(\Omega)$ *for some* $\varepsilon > 0$*, then*

$$\|y_\varrho - \overline{y}\|_{L^2(\Gamma)} \leq c\,\varrho\,\|\overline{y}\|_{H^1(\Gamma)}, \quad \|y_\varrho - \overline{y}_e\|_{H^1(\Omega)} \leq c\,\varrho^{1/2}\,\|\overline{y}\|_{H^1(\Gamma)}. \tag{8}$$

Here, the constant c depends on the mapping properties of the partial differential operator $I - \Delta$ *in certain Sobolev spaces, and the trace theorem.*

3 Conforming FE discretization on tensor product meshes

For a conforming fe discretization of the variational formulation (5), we need to introduce a fe space $Y_h \subset Y$ of functions with zero normal derivatives. In this paper, we restrict our considerations to the unit square $\Omega = (0,1)^d$ which allows us to use appropriate tensor product fe spaces. Let $\widetilde{S}_h^1(0,1) := \text{span}\{\varphi_i\}_{i=1}^{n-1}$ be the modified fe space of piecewise linear and continuous basis functions φ_i which are defined with respect to a decomposition $0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1$ of the unit interval, with the local mesh sizes $h_i := x_i - x_{i-1}$, $i = 1, \ldots, n$, and with the global mesh size $h := \max_i h_i$. While the basis functions φ_i for $i = 2, \ldots, n-2$ are the standard piecewise linear and continuous ones, the basis functions φ_1 and φ_{n-1} are 1 in the intervals (x_0, x_1) and (x_{n-1}, x_n), respectively; see Fig. 1.

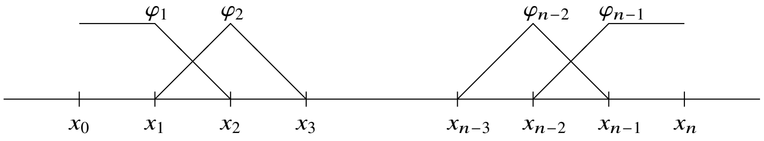

Fig. 1 Modified piecewise linear basis functions $\varphi_i(x)$, $i = 1, \ldots, n-1$.

By construction, we have $\varphi_i'(x) = 0$ for $x \in \{0,1\}$, $i = 1, \ldots, n-1$. We now define the conforming fe space $Y_h = \otimes_{i=1}^d \widetilde{S}_h^1(0,1) = \text{span}\{\phi_k\}_{k=1}^m \subset Y$ of piecewise multilinear continuous basis functions ϕ_k with vanishing Neumann trace $\partial_n \phi_k$ on Γ. We note that $m = (n-1)^d$. The fe discretization of the variational formulation (5) leads to the finite element scheme: Find $y_{\varrho h} \in Y_h$ such that

$$\langle y_{\varrho h}, y_h\rangle_{L^2(\Gamma)} + \varrho\,\langle y_{\varrho h}, y_h\rangle_{H^1(\Omega)} = \langle \overline{y}, y_h\rangle_{L^2(\Gamma)} \quad \text{for all } y_h \in Y_h. \tag{9}$$

Using standard arguments, we immediately arrive at the Cea-type estimate

$$\begin{aligned}\|y_\varrho - y_{\varrho h}\|^2_{L^2(\Gamma)} + \varrho\, \|y_\varrho - y_{\varrho h}\|^2_{H^1(\Omega)} & \\ \leq \|y_\varrho - y_h\|^2_{L^2(\Gamma)} + \varrho\, \|y_\varrho - y_h\|^2_{H^1(\Omega)} & \quad \text{for all } y_h \in Y_h.\end{aligned} \tag{10}$$

Lemma 2 *Let $y_{\varrho h} \in Y_h$ be the unique solution of* (9)*. Then, for $\overline{y} \in L^2(\Gamma)$, there holds the error estimate*

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq (1 + \sqrt{2})\, \|\overline{y}\|_{L^2(\Gamma)}. \tag{11}$$

If $\overline{y} \in H^{1/2}(\Gamma)$ is the Dirichlet trace of $\overline{y}_e \in Y$, and choosing $\varrho = h$, we have

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq c\, h^{1/2}\, \|\overline{y}\|_{H^{1/2}(\Gamma)}. \tag{12}$$

For $\overline{y} = \overline{y}_{e|\Gamma} \in H^1(\Gamma)$, $\overline{y}_e \in H^{3/2+\varepsilon}(\Omega)$ with $\varepsilon > 0$, and choosing $\varrho = h$, we have

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq c\, h\, \|\overline{y}\|_{H^1(\Gamma)}. \tag{13}$$

Proof. For $\overline{y} \in L^2(\Gamma)$, we consider (10) for $y_h = 0$, and we use (5) and (6) to obtain

$$\|y_\varrho - y_{\varrho h}\|^2_{L^2(\Gamma)} \leq \|y_\varrho\|^2_{L^2(\Gamma)} + \varrho\, \|y_\varrho\|^2_{H^1(\Omega)} \leq 2\, \|\overline{y}\|^2_{L^2(\Gamma)}.$$

With the triangle inequality and again using (6), we therefore conclude

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq \|y_{\varrho h} - y_\varrho\|_{L^2(\Gamma)} + \|y_\varrho - \overline{y}\|_{L^2(\Gamma)} \leq (1 + \sqrt{2})\, \|\overline{y}\|_{L^2(\Gamma)}\,.$$

For $\overline{y} = \overline{y}_{e|\Gamma} \in H^{1/2}(\Gamma)$ we use the triangle inequality twice and Cea's estimate (10) for arbitrary $y_h \in Y_h$ to write

$$\begin{aligned}\|y_{\varrho h} - \overline{y}\|^2_{L^2(\Gamma)} &\leq 2\, \|y_\varrho - \overline{y}\|^2_{L^2(\Gamma)} + 2\, \|y_\varrho - y_{\varrho h}\|^2_{L^2(\Gamma)} \\ &\leq 2\, \|y_\varrho - \overline{y}\|^2_{L^2(\Gamma)} + 2\, \|y_\varrho - y_h\|^2_{L^2(\Gamma)} + 2\, \varrho\, \|y_\varrho - y_h\|^2_{H^1(\Omega)} \\ &\leq 6\, \|y_\varrho - \overline{y}\|^2_{L^2(\Gamma)} + 4\, \|\overline{y} - y_h\|^2_{L^2(\Gamma)} \\ &\qquad + 4\, \varrho\, \|y_\varrho - \overline{y}_e\|^2_{H^1(\Omega)} + 4\, \varrho\, \|\overline{y}_e - y_h\|^2_{H^1(\Omega)}.\end{aligned} \tag{14}$$

In particular, for $y_h = P_h \overline{y}_e \in Y_h$ being the L^2 projection of $\overline{y}_e$, we have the standard fe error estimates; see, e.g., [8],

$$\|\overline{y}_e - P_h \overline{y}_e\|_{H^1(\Omega)} \leq c\, \|\overline{y}_e\|_{H^1(\Omega)}, \quad \|\overline{y}_e - P_h \overline{y}_e\|_{L^2(\Omega)} \leq c\, h\, \|\overline{y}_e\|_{H^1(\Omega)}. \tag{15}$$

When using [1, Theorem 3.6] and a space interpolation argument, we also have

$$\|\overline{y}_e - P_h \overline{y}_e\|_{L^2(\Gamma)} \leq c\, \|\overline{y}_e - P_h \overline{y}_e\|_{H^{1/2}(\Omega)} \leq c\, h^{1/2}\, \|\overline{y}_e\|_{H^1(\Omega)}.$$

Hence, using (7) and $\|\overline{y}_e\|_{H^1(\Omega)} \leq \|\overline{y}\|_{H^{1/2}(\Gamma)}$, this gives

$$\|y_{\varrho h} - \overline{y}\|^2_{L^2(\Gamma)} \ \leq\ c\, (\varrho + h)\, \|\overline{y}\|^2_{H^{1/2}(\Gamma)},$$

and (12) follows when choosing $\varrho = h$.

Finally, we consider the case $\overline{y} = \overline{y}_{e|\Gamma} \in H^1(\Gamma)$ with $\overline{y}_e \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. In this case, we have

$$\|\overline{y}_e - P_h\overline{y}_e\|_{H^1(\Omega)} \leq c\,h^{1/2}\,\|\overline{y}_e\|_{H^{3/2}(\Omega)},\ \|\overline{y}_e - P_h\overline{y}_e\|_{L^2(\Omega)} \leq c\,h^{3/2}\,\|\overline{y}_e\|_{H^{3/2}(\Omega)},$$

and $\|\overline{y} - P_h\overline{y}_e\|_{L^2(\Gamma)} \leq c\,h\,\|\overline{y}_e\|_{H^{3/2}(\Omega)}$. Together with (8) we therefore conclude $\|y_{\varrho h} - \overline{y}\|^2_{L^2(\Gamma)} \leq c\,(\varrho^2 + \varrho h + h^2)\,\|\overline{y}_e\|^2_{H^{3/2}(\Omega)}$, and, for $\varrho = h$, this gives (13). □

While the regularization error estimates as given in Lemma 1 are optimal in ϱ for $\overline{y} \in H^1(\Gamma)$, i.e., $\overline{y}_e \in H^{3/2}(\Omega)$, we can expect higher order convergence for the fe approximation $P_h\overline{y}_e$ when $\overline{y}_e$ is more regular. In particular for $\overline{y}_e \in H^2(\Omega)$ we have

$$\|\overline{y}_e - P_h\overline{y}_e\|_{H^1(\Omega)} \leq c\,h\,\|\overline{y}_e\|_{H^2(\Omega)},\ \|\overline{y}_e - P_h\overline{y}_e\|_{L^2(\Omega)} \leq c\,h^2\,\|\overline{y}_e\|_{H^2(\Omega)}.$$

Note that $\overline{y}_e \in H^2(\Omega)$ implies some additional compatibility conditions on $\overline{y}$, when $\Gamma = \partial\Omega$ is piecewise smooth. In particular, the tangential derivatives of $\overline{y}$ have to be in $H^{1/2}(\Gamma)$, see, e.g., the discussion in [2, 6] in the two-dimensional case. Then we can use (14) to conclude

$$\|y_{\varrho h} - \overline{y}\|^2_{L^2(\Gamma)} \leq c_1\,\varrho^2\,\|\overline{y}\|^2_{H^1(\Gamma)} + c_2\,(h^3 + \varrho\,h^2)\,\|\overline{y}_e\|^2_{H^2(\Omega)},$$

and for $\varrho \leq h^{3/2}$ we obtain

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq c\,h^{3/2}\left[\|\overline{y}_e\|^2_{H^2(\Omega)} + \|\overline{y}\|^2_{H^{3/2}(\Gamma)}\right]^{1/2}. \tag{16}$$

However, for $\overline{y} \in H^2(\Gamma)$, i.e., $\overline{y}_e \in H^{5/2}(\Omega)$, and using the best approximation estimate for the boundary term, this gives

$$\|y_{\varrho h} - \overline{y}\|^2_{L^2(\Gamma)} \leq c_1\,h^4\,\|\overline{y}\|^2_{H^2(\Gamma)} + c_2\,\varrho^2\,\|\overline{y}\|^2_{H^1(\Gamma)} + c_3\,\varrho\,h^2\,\|\overline{y}_e\|^2_{H^2(\Omega)},$$

and for $\varrho = h^2$ we finally obtain

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Gamma)} \leq c\,h^2\Big[\|\overline{y}\|^2_{H^2(\Gamma)} + \|\overline{y}\|^2_{H^1(\Gamma)} + \|\overline{y}_e\|^2_{H^2(\Omega)}\Big]^{1/2}. \tag{17}$$

4 Fast solvers

Once the basis is chosen, the finite element scheme (9) is equivalent to a linear system of finite element equations that can be written in the form

$$[\widetilde{\mathbf{M}}_h + \varrho(\widetilde{\mathbf{K}}_h + \mathring{\mathbf{K}}_h)]\mathbf{y}_h = \overline{\mathbf{y}}_h, \tag{18}$$

where the matrices $\widetilde{\mathbf{M}}_h$, $\widetilde{\mathbf{K}}_h$, and $\mathring{\mathbf{K}}_h$ have the respective block representations

$$\widetilde{\mathbf{M}}_h = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{BB} \end{pmatrix}, \quad \widetilde{\mathbf{K}}_h = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{K}}_{BB} \end{pmatrix}, \quad \text{and} \quad \mathring{\mathbf{K}}_h = \begin{pmatrix} \mathring{\mathbf{K}}_{II} & \mathring{\mathbf{K}}_{IB} \\ \mathring{\mathbf{K}}_{BI} & \mathring{\mathbf{K}}_{BB} \end{pmatrix},$$

when we split the unknowns (dofs) $\mathbf{y}_h = (\mathbf{y}_I^\top, \mathbf{y}_B^\top)^\top \in \mathbb{R}^{(n-1)^d}$ into strict interior unknowns $\mathbf{y}_I \in \mathbb{R}^{(n-3)^d}$ and near-boundary unknowns $\mathbf{y}_B \in \mathbb{R}^{(n-1)^d-(n-3)^d}$. The matrices $\mathbf{M}_{BB}$, $\widetilde{\mathbf{K}}_{BB}$ and $\mathring{\mathbf{K}}_h$ are defined by the identities $(\mathbf{M}_{BB}\mathbf{y}_B, \mathbf{v}_B) = \langle y_h, v_h \rangle_{L^2(\Gamma)}$, $(\widetilde{\mathbf{K}}_{BB}\mathbf{y}_B, \mathbf{v}_B) = \langle y_h, v_h \rangle_{H^1(\widetilde{\Omega}_h)}$, $(\mathring{\mathbf{K}}_h\mathbf{y}_h, \mathbf{v}_h) = \langle y_h, v_h \rangle_{H^1(\mathring{\Omega}_h)}$ for all $\mathbf{y}_h = (\mathbf{y}_I^\top, \mathbf{y}_B^\top)^\top \leftrightarrow y_h, v_h \in Y_h$ (fe isomorphism), where $\mathring{\Omega}_h = \Omega \setminus \overline{\widetilde{\Omega}}_h = (h, 1-h)^d$ and $\widetilde{\Omega}_h = \Omega \backslash \overline{\mathring{\Omega}}_h$, whereas $\overline{\mathbf{y}}_h = (\mathbf{0}_I^\top, \overline{\mathbf{y}}_B^\top)^\top \in \mathbb{R}^m$ is given by $(\overline{\mathbf{y}}_B, \mathbf{y}_B) = \langle \overline{y}_h, y_h \rangle_{L^2(\Gamma)}$ for all $\mathbf{y}_h = (\mathbf{y}_I^\top, \mathbf{y}_B^\top)^\top \leftrightarrow y_h, v_h \in Y_h$.

Eliminating $\mathbf{y}_I = -\mathring{\mathbf{K}}_{II}^{-1}\mathring{\mathbf{K}}_{IB}\mathbf{y}_B$ from the linear system (18), we arrive at the boundary Schur complement (SC) system

$$\mathbf{S}_{BB}\mathbf{y}_B = \overline{\mathbf{y}}_B \tag{19}$$

with $\mathbf{S}_{BB} = \mathbf{M}_{BB} + \varrho(\widetilde{\mathbf{K}}_{BB} + \mathring{\mathbf{S}}_{BB}) = \mathbf{M}_{BB} + \varrho(\widetilde{\mathbf{K}}_{BB} + (\mathring{\mathbf{K}}_{BB} - \mathring{\mathbf{K}}_{BI}\mathring{\mathbf{K}}_{II}^{-1}\mathring{\mathbf{K}}_{IB}))$. The Schur complement system (19) can efficiently be solved by means of the Conjugate Gradient (CG) method without any preconditioning since, for $\varrho \le h$, the Schur complement $\mathbf{S}_{BB}$ is spectrally equivalent to the boundary mass matrix $\mathbf{M}_{BB}$, and in turn $\mathbf{M}_{BB}$ is spectrally equivalent to the lumped boundary mass matrix $\text{lump}(\mathbf{M}_{BB})$ and to $h^{d-1}\mathbf{I}_{BB}$. Indeed, it is easy to see that

$$\mathbf{M}_{BB} \le \mathbf{S}_{BB} = \mathbf{M}_{BB} + \varrho(\widetilde{\mathbf{K}}_{BB} + \mathring{\mathbf{S}}_{BB}) \le (1 + \widetilde{c}\varrho h^{-1} + \mathring{c}\varrho h^{-1})\mathbf{M}_{BB} \tag{20}$$

with h and ϱ independent positive constants $\widetilde{c}$ and $\mathring{c}$ arising from the estimates $\lambda_{max}(\mathbf{M}_{BB}^{-1}\widetilde{\mathbf{K}}_{BB}) \le \widetilde{c}h^{-1}$ and $\lambda_{max}(\mathbf{M}_{BB}^{-1}\mathring{\mathbf{S}}_{BB}) \le \mathring{c}h^{-1}$ of the maximal eigenvalues of $\mathbf{M}_{BB}^{-1}\widetilde{\mathbf{K}}_{BB}$ and $\mathbf{M}_{BB}^{-1}\mathring{\mathbf{S}}_{BB}$, respectively. The choice $\varrho \le h$ delivers the desired result. It is recommended to use $\text{lump}(\mathbf{M}_{BB})$ as diagonal preconditioner in the Preconditioned Conjugate Gradient (PCG) method since it provides the right scaling. The numerical results presented in Section 5 show that the system (18) can also efficiently be solved by means of PCG with a simple Algebraic MultiGrid (AMG) preconditioner.

5 Numerical results

We first consider the target

$$\overline{y} = \overline{y}(x) := \cos(\pi x_1)\cos(\pi x_2)\cos(\pi x_3), \; x = (x_1, x_2, x_3) \in \Gamma, \tag{21}$$

on the boundary $\Gamma = \partial\Omega$ of the domain $\Omega = (0, 1)^3$. We mention that $\overline{y}$ is the trace of a smooth function with vanishing normal derivative on the boundary Γ, i.e., we have $\overline{y} \in H^2(\Gamma)$, and the error estimate (17) applies when choosing $\varrho = h^2$. We use a tensor product mesh as described in Section 3. The initial mesh contains 5 vertices in each direction, and 125 in total with mesh size $h = 0.25$. We note that we have only 3 dofs in each direction, and 27 in total for the initial level. Table 1 provides the numerical results starting from level $\ell = 1$ with 27 dofs and running to the finest discretization level $\ell = 7$ obtained by 6 uniform refinements of the initial mesh. The fourth column displays the L^2 error $\|y_\ell - \overline{y}\|_{L^2(\Gamma)}$ between the computed fe solution $y_\ell = y_{\varrho_\ell h_\ell}$ and the target $\overline{y}$ on the boundary. As expected, we observe second order of convergence; cf. experimental order of convergence (eoc) given in the fifth column. We first solve the original system (18) by means of AMG preconditioned CG iterations (#AMG-PCG its), and observe that not more than 4 iterations are needed in order to reach a relative residual error of 10^{-9}. We further test the CG and lumped mass preconditioned CG solvers for the Schur complement equation (19) until the relative residual error reaches 10^{-9}. The number of Schur complement CG (#SCG its) and lumped mass preconditioned CG (#SPCG its) iterations are displayed in the last two columns of Table 1. As expected from the theoretical results given in Section 4, we see level-independent iteration numbers in both cases. Moreover, the lumped-mass preconditioner further reduces the number of iteration by the scaling effect. We note that the action of $\mathring{\mathbf{K}}_{II}^{-1}$ to a vector within the multiplication of the Schur complement $\mathbf{S}_{BB}$ by some vector (iterate) is realized by an AMG preconditioned CG method until the relative residual error is reduced by a factor 10^{10}. The latter accuracy of this inner PCG iteration can be adapted (reduced !) to the outer CG/PCG iteration following the results from [7].

ℓ	#dofs	h	error	eoc	#AMG-PCG its	#SCG its	#PSCG its
1	27	2^{-2}	1.669e−1	–	2	1	1
2	343	2^{-3}	5.215e−2	1.68	3	6	6
3	3,375	2^{-4}	1.347e−2	1.95	4	17	9
4	29,791	2^{-5}	3.226e−3	2.06	4	24	9
5	250,047	2^{-6}	7.707e−4	2.07	4	28	8
6	2,048,383	2^{-7}	1.872e−4	2.04	4	29	6
7	16,581,375	2^{-8}	4.605e−5	2.02	4	29	4

Table 1 Target (21): error = $\|y_\ell - \overline{y}\|_{L^2(\Gamma)}$, eoc = $\log_2(\|y_\ell - \overline{y}\|_{L^2(\Gamma)}/\|y_{\ell+1} - \overline{y}\|_{L^2(\Gamma)})$, number of AMG CG iterations (#AMG-PCG its) for the original system (18), and number of CG (#SCG its) and lumped-mass preconditioned CG (#PSCG its) iterations for the SC system (19), $\varrho = h^2$.

The second target

$$\overline{y}(x) := x_1^2 - 0.5x_2^2 - 0.5x_3^2,\ x = (x_1, x_2, x_3) \in \Gamma, \tag{22}$$

is a trace of a smooth function which does not fulfill the homogeneous Neumann boundary conditions on the boundary Γ of $\Omega = (0,1)^3$, i.e., $\overline{y} \in H^{3/2-\varepsilon}(\Gamma)$, and $\overline{y}_e \in H^{2-\varepsilon}(\Omega)$, $\varepsilon > 0$. Similar as in (16) we therefore expect a reduced eoc of about 1.5. We perform the same tests for the target (22) as for the previous example for the target (21). The results are given in Table 2, where we have used $\varrho = h^{3/2}$ as prescribed by the theory. The iteration numbers are again independent of h, and show the same behavior as in the case of the first example.

ℓ	#Dofs	h	error	eoc	#AMG-PCG its	#SCG its	#PSCG its
1	27	2^{-2}	4.869e−1	–	2	5	5
2	343	2^{-3}	2.131e−1	1.19	3	13	10
3	3, 375	2^{-4}	8.177e−2	1.38	5	14	8
4	29, 791	2^{-5}	2.946e−2	1.47	4	15	8
5	250, 047	2^{-6}	1.036e−2	1.51	4	16	7
6	2, 048, 383	2^{-7}	3.619e−3	1.52	4	18	7
7	16, 581, 375	2^{-8}	1.265e−3	1.52	4	19	7

Table 2 Target (22): Same agenda as Table 1, but now $\varrho = h^{3/2}$.

6 Conclusions and Outlook

We have investigated a boundary value tracking, distributed ocp for some elliptic bvp with homogeneous Neumann boundary conditions. This ocp can be reduced to a state-based variational problem for functions from $H^1(\Omega)$ with vanishing normal derivatives in $H^{-1/2}(\Gamma)$. We have constructed a conforming fe discretization for the special case of tensor-product meshes, and we have derived discretization error estimates and fast solvers. The numerical experiments illustrate the theoretical results quantitatively. In general, one has to include the homogeneous Neumann boundary conditions by means of Lagrange multipliers, which is a topic of future research.

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References

1. Behrndt, J., Gesztesy, F., Mitrea, M.: Sharp boundary trace theory and Schrödinger operators on bounded Lipschitz domains. Mem. Amer. Math. Soc. **307**(1550), vi+208 (2025)
2. Jakovlev, G.N.: Boundary properties of functions of the class $W_p^{(l)}$ in regions with corners. Dokl. Akad. Nauk SSSR **140**, 73–76 (1961)
3. Langer, U., Löscher, R., Steinbach, O., Yang, H.: State-based nested iteration solution of a class of optimal control problems with PDE constraints. Math. Control Relat. Fields **15**, 1496–1537 (2025)
4. Mardal, K.A., Nielsen, B.F., Nordaas, M.: Robust preconditioners for PDE-constrained optimization with limited observations. BIT Numer. Math. **57**, 405–431 (2017)
5. Mardal, K.A., Sogn, J., Takacs, S.: Robust preconditioning and error estimates for optimal control of the convection–diffusion–reaction equation with limited observation in isogeometric analysis. SIAM J. Numer. Anal. **60**(1), 195–221 (2022)
6. Schmidt, G., Khoromskij, B.N.: Boundary integral equations for the biharmonic Dirichlet problem on nonsmooth domains. J. Integral Equations Appl. **11**, 217–253 (1999)
7. Simoncini, V., Szyld, D.: Theory of inexact Krylov subspace methods and application to scientific computing. SIAM J. Sci. Comput. **25**(2), 454–477 (2003)
8. Steinbach, O.: Numerical approximation methods for elliptic boundary value problems. Finite and boundary elements. Springer, New York (2008)