Convergence of Overlapping Schwarz within the Finite Volume Framework

Martin J. Gander[0000-0001-8450-9223],
Laurence Halpern[0000-0002-7877-7130],
Florence Hubert[0000-0002-7553-9698],
Stella Krell[0009-0006-2206-0882]

1 Overlapping Schwarz with Two Point Flux Approximation Finite Volume Discretization

We are interested in proving convergence of alternating Schwarz algorithms discretized with Two-Point Flux Approximation (TPFA) finite volume schemes. We use as our model problem the Poisson equation in a domain $\Omega \subset \mathbb{R}^2$ with boundary D, with a right hand side $f \in L^2(\Omega)$ and a Dirichlet boundary condition $g \in L^2(D)$. We decompose the domain Ω into two subdomains[1] Ω_1 and Ω_2 with overlap $\Omega_{12} = \Omega_{21} := \Omega_1 \cap \Omega_2$, and set $\Omega_{11} := \Omega_1 \setminus \overline{\Omega_{12}}$ and $\Omega_{22} := \Omega_2 \setminus \overline{\Omega_{21}}$, see Fig. 1 (left). The subdomain boundaries contain a Dirichlet part $D_i = D \cap \overline{\Omega_i}$, and an interface part Γ_i, with $\partial\Omega_i = D_i \cup \Gamma_i$. We use the standard notation of TPFA finite volume schemes [1], based on the use of admissible meshes $\mathfrak{M}$, see Fig. 1 (right), to ensure the necessary geometrical conditions for convergence under mesh refinement. We denote a generic control volume by K and m_K its measure, an edge connecting two generic control volumes by σ and m_σ its length, and the set of all edges by $\mathcal{E}$, see [1] for more details. We mesh Ω_{ij} and Ω_{ii} with admissible meshes $\mathfrak{M}_{ij}$ and $\mathfrak{M}_{ii}$, and let $\mathfrak{M}_i$ be the union of $\mathfrak{M}_{ij}$ of Ω_{ij} and $\mathfrak{M}_{ii}$ of Ω_{ii}. For an admissible mesh $\mathfrak{M}_i$, we set $\mathcal{T}_i := (\mathfrak{M}_i, \mathcal{E}_{D_i}, \mathcal{E}_{\Gamma_i})$ and the discrete approximation

---

Martin J. Gander
Section de Mathématiques, Université de Genève, Switzerland, e-mail: martin.gander@unige.ch

Laurence Halpern
LAGA, Université Sorbonne Paris-Nord, France, e-mail: halpern@math.univ-paris13.fr

Florence Hubert
Aix Marseille Univ, CNRS, I2M, Marseille, France, e-mail: florence.hubert@univ-amu.fr

Stella Krell
Université Côte d'Azur, CNRS, LJAD, France, e-mail: stella.krell@univ-cotedazur.fr

[1] The generalization to the many subdomain case does not pose any difficulty, since our discrete proofs are based on the classical techniques used by Schwarz and Lions at the continous level, which can be extended naturally to many subdomains.

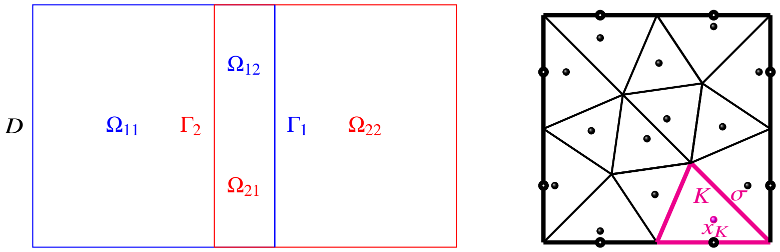


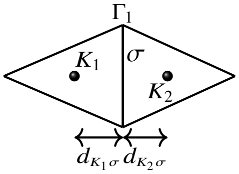
Fig. 1 Left: domain decomposition of Ω with boundary D into Ω_1 (blue) and Ω_2 (red). Right: admissible mesh with a generic control volume K and center x_K.

$u_{\mathcal{T}_i} := \left(u_{\mathfrak{M}_i}, u_{\mathcal{E}_{D_i}}, u_{\mathcal{E}_{\Gamma_i}}\right) = \left((u_K)_{K\in\mathfrak{M}_i}, (u_\sigma)_{\sigma\in\mathcal{E}_{D_i}}, (u_\sigma)_{\sigma\in\mathcal{E}_{\Gamma_i}}\right) \in \mathbb{R}^{\mathcal{T}_i}$ for $i = 1, 2$. Throughout the following, the notation $\mathbb{R}^{\mathfrak{M}}$ represents a constant vector on the mesh $\mathfrak{M}$. We initialize the discrete alternating Schwarz algorithm with some $g^1_{\mathcal{E}_{\Gamma_1}}$, and then it solves at iteration $\ell \geq 1$ on subdomain Ω_i the discrete problems

$$\mathcal{L}^{\mathcal{T}_i}(u^\ell_{i,\mathcal{T}_i}, f_{\mathfrak{M}_i}, g_{\mathcal{E}_{D_i}}, g^\ell_{\mathcal{E}_{\Gamma_i}}) = 0 \text{ with } g^{\ell+1}_{\mathcal{E}_{\Gamma_i}} = \begin{cases} P_{\mathcal{E}_{\Gamma_1}}(u^\ell_{2,\mathfrak{M}_2}) \text{ if } i = 1, \\ P_{\mathcal{E}_{\Gamma_2}}(u^{\ell+1}_{1,\mathfrak{M}_1}) \text{ if } i = 2, \end{cases} \tag{1}$$

with the averaging operators at the interfaces between two control volumes

$$P_\sigma(u_{\mathfrak{M}_i}) = \frac{u_{K_i} d_{K_j\sigma} + u_{K_j} d_{K_i\sigma}}{d_{K_i\sigma} + d_{K_j\sigma}} \tag{2}$$
$$\forall \sigma = K_i|K_j \in \mathcal{E}_{\Gamma_i},\ i \neq j \text{ and } K_i \in \mathfrak{M}_i,$$



and where we introduced the discrete operators

$$\mathcal{L}^{\mathcal{T}_i}(u_{\mathcal{T}_i}, f_{\mathfrak{M}_i}, g_{\mathcal{E}_{D_i}}, g^{\Gamma_i}_{\mathcal{E}_{\Gamma_i}}) = 0 \iff \begin{cases} \forall K \in \mathfrak{M}_i, \displaystyle\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u_{\mathcal{T}_i}) = m_K f_K, \\ \forall \sigma \in \mathcal{E}_{D_i},\ u_\sigma = g_\sigma,\ \forall \sigma \in \mathcal{E}_{\Gamma_i},\ u_\sigma = g^{\Gamma_i}_\sigma. \end{cases}$$

Here the fluxes are defined using the TPFA scheme, namely

$$F_{K,\sigma}(u_{\mathcal{T}}) := m_\sigma \frac{u_K - u_L}{d_{K\sigma} + d_{L\sigma}} \text{ if } \sigma = K|L \in \mathcal{E}_{int}, \text{ or } m_\sigma \frac{u_K - u_\sigma}{d_{K\sigma}} \text{ if } \sigma \in \mathcal{E}_{D_i} \cup \mathcal{E}_{\Gamma_i},$$

with $\mathcal{E}_{i,int} = \mathcal{E}_i \setminus \mathcal{E}_{D_i} \cup \mathcal{E}_{\Gamma_i}$ the internal edges. The source and boundary terms are $f_K := \frac{1}{m_K}\int_K f(x)dx$ for $K \in \mathfrak{M}_i$, and $g_\sigma := \frac{1}{m_\sigma}\int_\sigma g(x)dx$ for $\sigma \in \mathcal{E}_{D_i} \cup \mathcal{E}_{\Gamma_i}$. If the meshes coincide in the overlap, we can define the composite mesh $\mathfrak{M} := \mathfrak{M}_{11} \cup \mathfrak{M}_{12} \cup \mathfrak{M}_{22}$ on Ω and $u_{\mathfrak{M}} := (u_{\mathfrak{M}_{11}}, u_{\mathfrak{M}_{12}}, u_{\mathfrak{M}_{22}}) \in \mathbb{R}^{\mathfrak{M}}$, and $u_{\mathcal{T}} := (u_{\mathfrak{M}}, g_{\mathcal{E}_D}) \in \mathbb{R}^{\mathcal{T}}$ is discrete harmonic in Ω with data g on the boundary,

$$\mathcal{L}^{\mathcal{T}}(u_{\mathcal{T}}, f_{\mathfrak{M}}, g_{\mathcal{E}_D}) = 0. \tag{3}$$

Theorem 1 *The iterates* $u^\ell_{i,\mathfrak{M}_i}$ *of the discrete overlapping Schwarz algorithm* (1) *converge to* $u_{i,\mathfrak{M}_i}$ *when* ℓ *tends to* ∞*. In the case where the control volumes of* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *coincide in the overlap, the limit is* $u_{\mathfrak{M}}$ *solution of* (3).

We will give two different proofs of convergence: the first proof in Section 2 uses the maximum principle with $f \equiv 0$ like in the original proof of Schwarz in [4] at the continuous level, and the second proof in Section 3 with $g \equiv 0$ is based on a projection argument like Lions' proof at the continuous level in [3]. Note that in both cases, one could also argue directly with the errors $u_{i,\mathfrak{M}_i} - u^\ell_{i,\mathfrak{M}_i}$ to remove the hypotheses $f \equiv 0$ and $g \equiv 0$, so these are without loss of generality. For a discretization by a Discrete Duality Finite Volume (DDFV) method, a projection argument was also used in [2] to prove convergence of a discrete overlapping Schwarz method. Similarly to many numerical discretizations defined on general meshes, the DDFV method does not satisfy a discrete maximum principle.

2 Convergence using the maximum principle

In this section, the source f is zero, and we will consider discrete harmonic $u_{\mathcal{T}}$. We initialize the Schwarz algorithm with $g^1_{\mathcal{E}_{\Gamma_1}} = \underline{g} := \inf_D g$ where g is the Dirichlet boundary condition, and we also define $\overline{g} := \sup_D g$. We start by giving two results on the discrete maximum principle in the context of TPFA, and then prove convergence. The proof of Propositions 2 and 3 are directly obtained by the maximum principle given in Proposition 1, see also [1][Proposition 3.2]. The results are stated for the particular configuration in Fig. 1, and the inequalities are understood componentwise and hold on each cell.

Proposition 1 *For* $j = 1, 2$*, the discrete harmonic function in* Ω_j *with data equal to* g_σ *on* $\partial\Omega_j$ *reaches its extrema on the boundary, and satisfies* $\min_{\sigma\in\mathcal{E}_{\partial\Omega_j}} g_\sigma \le u_{\mathfrak{M}_j} \le \max_{\sigma\in\mathcal{E}_{\partial\Omega_j}} g_\sigma$.

Proposition 2 *For* $j = 1, 2$*, if* $g_{1,\mathcal{E}_{\Gamma_j}} \le g_{2,\mathcal{E}_{\Gamma_j}}$*, the discrete harmonic functions in* Ω_j *with data equal to 0 on* D_j *and* $g_{i,\mathcal{E}_{\Gamma_j}}$ *on* Γ_j *satisfy* $u_{1,\mathfrak{M}_j} \le u_{2,\mathfrak{M}_j}$.

Proposition 3 *For* $j = 1, 2$*, there exists a constant* $q_j \in (0, 1)$ *such that the discrete harmonic function which equals 0 on* D_j *and* 1 *on* Γ_j *satisfies* $0 \le P_{\mathcal{E}_{\Gamma_i}}(u_{j,\mathfrak{M}_j}) \le q_j$ *for* $i \neq j$ *with* P_σ *defined in* (2).

Proof. By the maximum principle in Proposition 1, $u_{j,\mathfrak{M}_j} \in (0, 1)$, therefore by definition $P_{\mathcal{E}_{\Gamma_i}}(u_{j,\mathfrak{M}_j}) \in (0, 1)$. Its maximum over all $\sigma \in \mathcal{E}_{\Gamma_i}$ is called q_j. □

Proof of Theorem 1: We start the algorithm in Ω_1 with the discrete harmonic function $u^1_{1,\mathfrak{M}_1}$ equal to g on D_1 and $\underline{g}$ on Γ_1. By Proposition 1, $\underline{g} \le u^1_{1,\mathfrak{M}_1} \le \overline{g}$, and by definition $\underline{g} \le P_{\mathcal{E}_{\Gamma_2}}(u^1_{1,\mathfrak{M}_1}) \le \overline{g}$. The algorithm then solves in Ω_2 with data g on D_2 and $g^1_{\mathcal{E}_{\Gamma_2}} = P_{\mathcal{E}_{\Gamma_2}}(u^1_{1,\mathfrak{M}_1})$ on Γ_2. Again $\underline{g} \le u^1_{2,\mathfrak{M}_2} \le \overline{g}$, and $\underline{g} \le g^1_{\mathcal{E}_{\Gamma_1}} = P_{\mathcal{E}_{\Gamma_1}}(u^1_{2,\mathfrak{M}_2}) \le \overline{g}$.

At the second iteration we also have $\underline{g} \leq u^2_{1,\mathfrak{M}_1} \leq \overline{g}$. The difference $u^2_{1,\mathfrak{M}_1} - u^1_{1,\mathfrak{M}_1}$ is discrete harmonic in Ω_1, with zero data on D_1, and data on Γ_1 equal to

$$P_{\mathcal{E}_{\Gamma_1}}(u^1_{2,\mathfrak{M}_2}) - \underline{g} \in [0, G], \quad G := \overline{g} - \underline{g}.$$

This implies that $u^2_{1,\mathfrak{M}_1} - u^1_{1,\mathfrak{M}_1} \in [0, G]$. Propositions 2 and 3 now imply that $0 \leq P_{\mathcal{E}_{\Gamma_2}}(u^2_{1,\mathfrak{M}_1} - u^1_{1,\mathfrak{M}_1}) \leq q_1 G$. Proceeding similarly in Ω_2, we obtain $\underline{g} \leq u^2_{2,\mathfrak{M}_2} \leq \overline{g}$, with $u^2_{2,\mathfrak{M}_2} - u^1_{2,\mathfrak{M}_2}$ discrete harmonic in Ω_2, with zero data on D_1, and data on Γ_2 equal to $P_{\mathcal{E}_{\Gamma_2}}(u^2_{1,\mathfrak{M}_1} - u^1_{1,\mathfrak{M}_1}) \in [0, q_1 G]$. Following the discrete Schwarz algorithm thus creates the sequence $u^\ell_{j,\mathfrak{M}_j} \in [\underline{g}, \overline{g}]$ with

$$0 < u^{\ell+1}_{1,\mathfrak{M}_1} - u^\ell_{1,\mathfrak{M}_1} \leq (q_1 q_2)^{\ell-1} G, \quad 0 < u^{\ell+1}_{2,\mathfrak{M}_2} - u^\ell_{2,\mathfrak{M}_2} \leq q_1 (q_1 q_2)^{\ell-1} G.$$

Writing $0 \leq u^{\ell+1}_{i,\mathfrak{M}_i} - u^1_{i,\mathfrak{M}_i} = \sum_{p=1}^{\ell} (u^{p+1}_{i,\mathfrak{M}_i} - u^p_{i,\mathfrak{M}_i})$, the terms in the sum are bounded by $G(q_1 q_2)^{p-1}$, therefore the series is convergent, defining a limit $u_{i,\mathfrak{M}_i}$. Furthermore, by continuity, $g^\ell_{\mathcal{E}_{\Gamma_i}}$ converges to $P_{\mathcal{E}_{\Gamma_i}}(u_{j,\mathfrak{M}_j}) := g_{\mathcal{E}_{\Gamma_i}}$. Now, if the meshes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are identical on the overlap, denoted by $\mathfrak{M}_{12}$, we define in Ω_{12} $u_{i,\mathfrak{M}_{12}}$ as $u_{i,\mathfrak{M}_i}$ in the volume, g on D_i, $g_{\mathcal{E}_{\Gamma_i}}$ on Γ_i, and $P_{\mathcal{E}_{\Gamma_j}}(u_{i,\mathfrak{M}_i})$ on Γ_j. The difference is discrete harmonic in the overlap, with data 0 on D, and on Γ_i, $g_{\mathcal{E}_{\Gamma_i}} - P_{\mathcal{E}_{\Gamma_i}}(u_{j,\mathfrak{M}_j})$ which is zero, by passing to the limit in (1). By the maximum principle the difference is zero, the limits coincide in the overlap. Thus, the limit $u_{\mathfrak{M}} = (u_{1,\mathfrak{M}_{11}}, u_{1,\mathfrak{M}_{12}}, u_{2,\mathfrak{M}_{22}})$ is solution of (3).

3 Convergence using projections

We now assume, again without loss of generality as explained earlier, that the Dirichlet boundary condition is homogeneous, $g = 0$, and that in the overlap the meshes coincide, $\mathfrak{M}_{12} = \mathfrak{M}_{12}$, and prove convergence using projection operators. We initialize the algorithm with $g^1_{\mathcal{E}_{\Gamma_1}} = 0$. We define on $\mathbb{R}^{\mathfrak{M}}$ the bilinear and linear forms

$$a_h^{\mathfrak{M}}(w_{\mathfrak{M}}, v_{\mathfrak{M}}) := \sum_{\sigma \in \mathcal{E}} \frac{m_\sigma}{d_\sigma} D_\sigma(w_{\mathfrak{M}}) D_\sigma(v_{\mathfrak{M}}), \quad L_h(v_{\mathfrak{M}}) := \sum_{K \in \mathfrak{M}} m_K f_K v_K,$$

where the discrete gradients are given by

$$D_\sigma(v_{\mathfrak{M}}) := v_L - v_K \text{ if } \sigma = K|L \in \mathcal{E}_{int}, \text{ or } -v_K \text{ if } \sigma \in \mathcal{E}_D.$$

In the following, the same notation L_h is kept for $L_h(v_{\mathfrak{M}_i})$, even if $v_{\mathfrak{M}_i} \in \mathbb{R}^{\mathfrak{M}_i}$; in this case, the sum will be taken over M_i only, instead of $\mathfrak{M}$. The variational formulation of (3) on the entire domain is $u_{\mathcal{T}} = (u_{\mathfrak{M}}, 0_{\mathcal{E}_D}) \in \mathbb{R}^{\mathcal{T}}$ and

$$\text{Find } u_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}} \text{ s. t. } a_h^{\mathfrak{M}}(u_{\mathfrak{M}}, v_{\mathfrak{M}}) = L_h(v_{\mathfrak{M}}), \quad \forall v_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}}. \tag{4}$$

In order to include the Dirichlet traces for the transmission conditions in the Schwarz algorithm (1) in the variational formulation, we extend $v_{\mathfrak{M}}$ by adding the values on the two interfaces, $v_{\widetilde{\mathcal{T}}} = (v_{\mathfrak{M}_{11}}, v_{\mathcal{E}_{\Gamma_2}}, v_{\mathfrak{M}_{12}}, v_{\mathcal{E}_{\Gamma_1}}, v_{\mathfrak{M}_{22}}) \in \mathbb{R}^{\widetilde{\mathcal{T}}}$. With these interface values, we obtain on the extended $\mathbb{R}^{\widetilde{\mathcal{T}}}$ the bilinear form

$$\begin{aligned} a_h^{\widetilde{\mathcal{T}}}(w_{\widetilde{\mathcal{T}}}, v_{\widetilde{\mathcal{T}}}) := & \sum_{\sigma \in \mathcal{E} \setminus \{\mathcal{E}_{\Gamma_1} \cup \mathcal{E}_{\Gamma_2}\}} \frac{m_\sigma}{d_\sigma} D_\sigma(w_{\mathfrak{M}}) D_\sigma(v_{\mathfrak{M}}) \\ & + \sum_{\sigma = K_1 | K_2 \in \mathcal{E}_{\Gamma_1} \cup \mathcal{E}_{\Gamma_2}} \sum_{k=1}^{2} \frac{m_\sigma}{d_{K_k \sigma}} (w_\sigma - w_{K_k})(v_\sigma - v_{K_k}). \end{aligned} \tag{5}$$

We define on the subdomains $\mathbb{R}^{\mathcal{T}_i}$ the bilinear forms

$$a_{h,i}(w_{\mathcal{T}_i}, v_{\mathcal{T}_i}) := \sum_{\sigma \in \mathcal{E}_i \setminus \mathcal{E}_{\Gamma_i}} \frac{m_\sigma}{d_\sigma} D_\sigma(w_{\mathfrak{M}_i}) D_\sigma(v_{\mathfrak{M}_i}) + \sum_{\sigma = K | \Gamma_i \in \mathcal{E}_{\Gamma_i}} \frac{m_\sigma}{d_{K\sigma}} (w_\sigma - w_K)(v_\sigma - v_K).$$

The variational formulation on each subdomain Ω_i of (1) is then

$$\begin{aligned} & \text{Find } u_{i,\mathcal{T}_i}^{\ell} = (u_{i,\mathfrak{M}_i}^{\ell}, 0_{\mathcal{E}_{D_i}}, g_{\mathcal{E}_{\Gamma_i}}^{\ell}) \in \mathbb{R}^{\mathcal{T}_i} \text{ s. t.} \\ & a_{h,i}(u_{i,\mathcal{T}_i}^{\ell}, v_{0,\mathcal{T}_i}) = L_h(v_{\mathfrak{M}_i}), \qquad \forall v_{0,\mathcal{T}_i} = (v_{\mathfrak{M}_i}, 0_{\mathcal{E}_{D_i}}, 0_{\mathcal{E}_{\Gamma_i}}) \in \mathbb{R}_0^{\mathcal{T}_i}. \end{aligned} \tag{6}$$

In order to rewrite (6) with the extended bilinear form $a_h^{\widetilde{\mathcal{T}}}$, we need an operator $P_{\widetilde{\mathcal{T}}}$ that transforms $v_{\mathfrak{M}}$ into a $v_{\widetilde{\mathcal{T}}}$ containing the traces on Γ_1 and Γ_2,

$$P_{\widetilde{\mathcal{T}}}(v_{\mathfrak{M}}) := (v_{\mathfrak{M}_{11}}, P_{\mathcal{E}_{\Gamma_2}}(v_{\mathfrak{M}}), v_{\mathfrak{M}_{12}}, P_{\mathcal{E}_{\Gamma_1}}(v_{\mathfrak{M}}), v_{\mathfrak{M}_{22}}) \in \mathbb{R}^{\widetilde{\mathcal{T}}}.$$

We also define for any $w_{\mathfrak{M}_2}$ an operator $P_{2\to1}^{w_{\mathfrak{M}_2}} : \mathbb{R}^{\mathfrak{M}_1} \to \mathbb{R}^{\widetilde{\mathcal{T}}}$ that extends $v_{\mathfrak{M}_1}$ with values from $w_{\mathfrak{M}_2}$ to $v_{\widetilde{\mathcal{T}}}$ on the entire domain,

$$P_{2\to1}^{w_{\mathfrak{M}_2}}(v_{\mathfrak{M}_1}) = (v_{\mathfrak{M}_{11}}, P_{\mathcal{E}_{\Gamma_2}}(v_{\mathfrak{M}_1}), v_{\mathfrak{M}_{12}}, P_{\mathcal{E}_{\Gamma_1}}(w_{\mathfrak{M}_2}), w_{\mathfrak{M}_{22}}),$$

and for any $w_{\mathfrak{M}_1}$ an operator $P_{1\to2}^{w_{\mathfrak{M}_1}} : \mathbb{R}^{\mathfrak{M}_2} \to \mathbb{R}^{\widetilde{\mathcal{T}}}$ that extends $v_{\mathfrak{M}_2}$ with values from $w_{\mathfrak{M}_1}$ to $v_{\widetilde{\mathcal{T}}}$ on the entire domain,

$$P_{1\to2}^{w_{\mathfrak{M}_1}}(v_{\mathfrak{M}_2}) = (w_{\mathfrak{M}_{11}}, P_{\mathcal{E}_{\Gamma_2}}(w_{\mathfrak{M}_1}), v_{\mathfrak{M}_{12}}, P_{\mathcal{E}_{\Gamma_1}}(v_{\mathfrak{M}_2}), v_{\mathfrak{M}_{22}}).$$

We define two subsets of $\mathbb{R}^{\widetilde{\mathcal{T}}}$ representing subdomain functions extended by zero, $V_1 := \{P_{2\to1}^{0_{\mathfrak{M}_2}}(v_{\mathfrak{M}_1}), v_{\mathfrak{M}_1} \in \mathbb{R}^{\mathfrak{M}_1}\}$ and $V_2 := \{P_{1\to2}^{0_{\mathfrak{M}_1}}(v_{\mathfrak{M}_2}),\ v_{\mathfrak{M}_2} \in \mathbb{R}^{\mathfrak{M}_2}\}$

$$\begin{aligned} v \in V_1 & \iff \exists v_{\mathfrak{M}_1},\ v = (v_{\mathfrak{M}_{11}}, P_{\mathcal{E}_{\Gamma_2}}(v_{\mathfrak{M}_1}), v_{\mathfrak{M}_{12}}, 0_{\mathcal{E}_{\Gamma_1}}, 0_{\mathfrak{M}_{22}}), \\ v \in V_2 & \iff \exists v_{\mathfrak{M}_2},\ v = (0_{\mathfrak{M}_{11}}, 0_{\mathcal{E}_{\Gamma_2}}, v_{\mathfrak{M}_{12}}, P_{\mathcal{E}_{\Gamma_1}}(w_{\mathfrak{M}_2}), w_{\mathfrak{M}_{22}}). \end{aligned} \tag{7}$$

Lemma 1 *$\forall (w_{\mathfrak{M}}, v_{\mathfrak{M}_i}) \in \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}_i}$, and $v_{0,\mathfrak{M}} := (v_{\mathfrak{M}_i}, 0_{\mathfrak{M}_{jj}}) \in \mathbb{R}^{\mathfrak{M}}$, we have*

$$a_h^{\widetilde{\mathcal{T}}}(P_{\widetilde{\mathcal{T}}}(w_{\mathfrak{M}}), P_{j\to i}^{0_{\mathfrak{M}_j}}(v_{\mathfrak{M}_i})) = a_h^{\mathfrak{M}}(w_{\mathfrak{M}}, v_{0,\mathfrak{M}}), \tag{8}$$

and $\forall (w_{\widetilde{\mathcal{T}}}, v_{\mathfrak{M}_i}) \in \mathbb{R}^{\widetilde{\mathcal{T}}} \times \mathbb{R}^{\mathfrak{M}_i}$, with $v_{0,\mathcal{T}_i} := (v_{\mathfrak{M}_i}, 0_{\mathcal{E}_{D_i}}, 0_{\mathcal{E}_{\Gamma_i}}) \in \mathbb{R}_0^{\mathcal{T}_i}$, we have

$$a_h^{\widetilde{\mathcal{T}}}(w_{\widetilde{\mathcal{T}}}, P_{j\to i}^{0_{\mathfrak{M}_j}}(v_{\mathfrak{M}_i})) = a_{h,i}(w_{\mathcal{T}_i}, v_{0,\mathcal{T}_i}). \tag{9}$$

Proof. To get equality (8), we only need to look at one interface Γ_i, for the other interface the argument is analogous due to the definition of P_σ. Setting $i = 1$, the contribution for every $\sigma = K_1|K_2 \in \mathcal{E}_{\Gamma_1}$ (with $K_1 \in \mathfrak{M}_{12}$ and $K_2 \in \mathfrak{M}_{22}$) on the left hand side of (8) is

$$\begin{aligned} T_L &:= \frac{(P_\sigma(w_{\mathfrak{M}}) - w_{K_1})(v_\sigma - v_{K_1})}{d_{K_1\sigma}} + \frac{(P_\sigma(w_{\mathfrak{M}}) - w_{K_2})(v_\sigma - v_{K_2})}{d_{K_2\sigma}} \\ &= -\frac{(P_\sigma(w_{\mathfrak{M}}) - w_{K_1})v_{K_1}}{d_{K_1\sigma}}, \end{aligned}$$

since $v_\sigma = v_{K_2} = 0$. On the right hand side of (8), we get

$$T_R := \frac{(w_{K_2} - w_{K_1})(v_{K_2} - v_{K_1})}{d_{K_1\sigma} + d_{K_2\sigma}} = -\frac{(w_{K_2} - w_{K_1})v_{K_1}}{d_{K_1\sigma} + d_{K_2\sigma}},$$

since $v_{K_2} = 0$. The definition (2) of $P_\sigma(w_{\mathfrak{M}})$ implies then after a short computation that $T_R = T_L$. Similarly, we can also prove (9). □

From the solution $u_{\mathfrak{M}}$ of (4) on the entire domain, we extend solutions in $\mathbb{R}^{\widetilde{\mathcal{T}}}$, in the sense of Lemma 1, including interface values,

$$\widetilde{u}_{1,\widetilde{\mathcal{T}}} := P_{2\to 1}^{u_{\mathfrak{M}_2}}(u_{\mathfrak{M}_1}) \quad \text{and} \quad \widetilde{u}_{2,\widetilde{\mathcal{T}}} := P_{1\to 2}^{u_{\mathfrak{M}_1}}(u_{\mathfrak{M}_2}).$$

Note that by definition, we have $\widetilde{u}_{1,\widetilde{\mathcal{T}}} = \widetilde{u}_{2,\widetilde{\mathcal{T}}} = P_{\widetilde{\mathcal{T}}}(u_{\mathfrak{M}})$. We define the orthogonal projections $\mathbb{P}_j := \mathbb{P}_{V_j}^{a_h^{\widetilde{\mathcal{T}}}}$ in $\mathbb{R}^{\widetilde{\mathcal{T}}}$ on V_j and $\mathbb{Q}_j := \mathbb{P}_{V_j}^{\perp a_h^{\widetilde{\mathcal{T}}}}$ in $\mathbb{R}^{\widetilde{\mathcal{T}}}$ on $V_j^{\perp}$, which are naturally related by $\mathbb{Q}_j = \mathbb{I} - \mathbb{P}_j$. From the solutions on the subdomains $(u_{1,\mathcal{T}_1}^{\ell}, u_{2,\mathcal{T}_2}^{\ell}) \in \mathbb{R}^{\mathcal{T}_1} \times \mathbb{R}^{\mathcal{T}_2}$ of (6) we define extended solutions on the entire domain in $\mathbb{R}^{\widetilde{\mathcal{T}}}$, in the sense of Lemma 1, using the previous iterate on the rest of the domain, namely

$$\widetilde{u}_{1,\widetilde{\mathcal{T}}}^{\ell} := P_{2\to 1}^{u_{2,\mathfrak{M}_2}^{\ell-1}}(u_{1,\mathfrak{M}_1}^{\ell}) \quad \text{and} \quad \widetilde{u}_{2,\widetilde{\mathcal{T}}}^{\ell} := P_{1\to 2}^{u_{1,\mathfrak{M}_1}^{\ell}}(u_{2,\mathfrak{M}_2}^{\ell}). \tag{10}$$

Lemma 2 *The extended solutions from* (10) *satisfy the projection identities*

$$\widetilde{u}_{1,\widetilde{\mathcal{T}}}^{\ell} - \widetilde{u}_{1,\widetilde{\mathcal{T}}} = \mathbb{Q}_1\left(\widetilde{u}_{2,\widetilde{\mathcal{T}}}^{\ell-1} - \widetilde{u}_{2,\widetilde{\mathcal{T}}}\right) \quad \textit{and} \quad \widetilde{u}_{2,\widetilde{\mathcal{T}}}^{\ell} - \widetilde{u}_{2,\widetilde{\mathcal{T}}} = \mathbb{Q}_2\left(\widetilde{u}_{1,\widetilde{\mathcal{T}}}^{\ell} - \widetilde{u}_{1,\widetilde{\mathcal{T}}}\right).$$

Proof. Using Lemma 1 for the solution of (4) and (6) implies that

$$\begin{aligned}a_h^{\widetilde{\mathcal{T}}}(\widetilde{u}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}}, P^{0_{\mathfrak{M}_2}}_{2\to1}(v_{\mathfrak{M}_1})) &= a_h^{\mathfrak{M}}(u_{\mathfrak{M}}, v_{0,\mathfrak{M}}) - a_{h,1}(u^{\ell}_{1,\mathcal{T}_1}, v_{0,\mathcal{T}_1})\\ &= L_h(v_{0,\mathfrak{M}}) - L_h(v_{\mathfrak{M}_1}) = 0.\end{aligned}$$

Now adding $\pm\widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}}$ in the first argument of the bilinear form gives

$$a_h^{\widetilde{\mathcal{T}}}(\widetilde{u}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} - (\widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}}), P^{0_{\mathfrak{M}_2}}_{2\to1}(v_{\mathfrak{M}_1})) = 0. \tag{11}$$

By definition, we have that $\widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} \in V_1$, which implies together with (11) the projection identity

$$\widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} = \mathbb{P}_1\left(\widetilde{u}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}}\right). \tag{12}$$

Adding $\widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} - \widetilde{u}_{1,\widetilde{\mathcal{T}}}$ on both sides, and using that $\mathbb{Q}_1 = \mathbb{I} - \mathbb{P}_1$ and $\widetilde{u}_{1,\widetilde{\mathcal{T}}} = \widetilde{u}_{2,\widetilde{\mathcal{T}}}$, we get

$$\widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}_{1,\widetilde{\mathcal{T}}} = \mathbb{Q}_1\left(\widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} - \widetilde{u}_{1,\widetilde{\mathcal{T}}}\right) = \mathbb{Q}_1\left(\widetilde{u}^{\ell-1}_{2,\widetilde{\mathcal{T}}} - \widetilde{u}_{2,\widetilde{\mathcal{T}}}\right).$$

The second identity can be proved similarly. □

Proof of Theorem 1: To simplify the notation, we introduce now the differences $\widetilde{v}^{\ell}_{1,\widetilde{\mathcal{T}}} := \widetilde{u}^{\ell}_{1,\widetilde{\mathcal{T}}} - \widetilde{u}_{1,\widetilde{\mathcal{T}}}$ and $\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} := \widetilde{u}^{\ell}_{2,\widetilde{\mathcal{T}}} - \widetilde{u}_{2,\widetilde{\mathcal{T}}}$. We thus have by Lemma 2 for all ℓ the projection identities

$$\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} = \mathbb{Q}_1\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}, \quad \widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}} = \mathbb{Q}_2\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}. \tag{13}$$

To prove convergence, we have to prove that the sequence of differences $\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}$ tends to 0. From (13), adding and subtracting $\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}$, we get

$$\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} = \widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} + \widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} = \mathbb{P}_1\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} + \mathbb{Q}_1\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}.$$

We now use the Pythagoras Theorem, and obtain the inequality

$$\|\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} = \|\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} + \|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} \geq \|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}}. \tag{14}$$

Similarly, also using (13) but now adding and subtracting $\widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}$, we get

$$\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} = \widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}} + \widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}} = \mathbb{P}_2\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} + \mathbb{Q}_2\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}},$$

which leads with the Pythagoras Theorem to the inequality

$$\|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} = \|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} + \|\widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} \geq \|\widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}}. \tag{15}$$

The two norm inequalities (14) and (15) then show that the two sequences $\|\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}\|_{a_h^{\widetilde{\mathcal{T}}}}$ and $\|\widetilde{v}^{\ell}_{1,\widetilde{\mathcal{T}}}\|_{a_h^{\widetilde{\mathcal{T}}}}$ are non-increasing. Since they are non-negative, they must converge to a limit, say L_2 and L_1. The inequalities (14) and (15) furthermore imply for these limits that $L_2 \geq L_1 \geq L_2$, and hence the limits must be equal, $L_1 = L_2 = L$.

Moreover, (14) implies that

$$\|\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} = \|\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} - \|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} \xrightarrow[\ell\to+\infty]{} 0, \tag{16}$$

and (15) implies that

$$\|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} = \|\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} - \|\widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}}\|^2_{a_h^{\widetilde{\mathcal{T}}}} \xrightarrow[\ell\to+\infty]{} 0.$$

It implies the convergence of $\widetilde{v}^{\ell+1}_{2,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}$ and $\widetilde{v}^{\ell+1}_{1,\widetilde{\mathcal{T}}} - \widetilde{v}^{\ell}_{1,\widetilde{\mathcal{T}}}$ to zero. Since $\|\widetilde{v}^{\ell}_{i,\widetilde{\mathcal{T}}}\|_{a_h^{\widetilde{\mathcal{T}}}}$ is bounded, there exists a sub-sequence $\varphi(\ell)$ that converges,

$$\widetilde{v}^{\varphi(\ell)}_{1,\widetilde{\mathcal{T}}} \xrightarrow[\ell\to+\infty]{} \widetilde{v}_{1,\widetilde{\mathcal{T}}} \qquad \widetilde{v}^{\varphi(\ell)}_{2,\widetilde{\mathcal{T}}} \xrightarrow[\ell\to+\infty]{} \widetilde{v}_{2,\widetilde{\mathcal{T}}}.$$

With (13), we have that $\widetilde{v}^{\varphi(\ell)}_{2,\widetilde{\mathcal{T}}} \in V_2^{\perp}$, which implies for all $v \in V_2$ that $a_h^{\widetilde{\mathcal{T}}}(\widetilde{v}^{\varphi(\ell)}_{2,\widetilde{\mathcal{T}}}, v) = 0$. Passing to the limit, we get for all $v \in V_2$ that $(a_h^{\widetilde{\mathcal{T}}}\widetilde{v}_{2,\widetilde{\mathcal{T}}}, v) = 0$, which implies that $\widetilde{v}_{2,\widetilde{\mathcal{T}}} \in V_2^{\perp}$. Since the projection $\mathbb{P}_1$ preserves convergence, we have by definition and (16) that

$$\mathbb{P}_1\widetilde{v}^{\varphi(\ell)}_{2,\widetilde{\mathcal{T}}} = \widetilde{v}^{\varphi(\ell)}_{2,\widetilde{\mathcal{T}}} - \widetilde{v}^{\varphi(\ell)+1}_{1,\widetilde{\mathcal{T}}} \xrightarrow[\ell\to+\infty]{} \mathbb{P}_1\widetilde{v}_{2,\widetilde{\mathcal{T}}} = 0.$$

This implies that $\widetilde{v}_{2,\widetilde{\mathcal{T}}} \in V_1^{\perp}$. So $\widetilde{v}_{2,\widetilde{\mathcal{T}}}$ must lie in the intersection, $\widetilde{v}_{2,\widetilde{\mathcal{T}}} \in V_1^{\perp} \cap V_2^{\perp}$. Now this intersection is $\{0\}$, by the definition in (7). Hence, $\widetilde{v}_{2,\widetilde{\mathcal{T}}} = 0$, and we have proved that the sequence $\widetilde{v}^{\ell}_{2,\widetilde{\mathcal{T}}}$ converges to 0. The result for $\widetilde{v}^{\ell}_{1,\widetilde{\mathcal{T}}}$ is obtained similarly.

References

1. Eymard, R., Gallouët, T., Herbin, R.: Finite Volume methods. In: P. Ciarlet, J. Lions (eds.) Handbook of numerical analysis, Vol. VII, Handb. Numer. Anal., VII, pp. 715–1022. North-Holland, Amsterdam (2000)
2. Gander, M.J., Halpern, L., Hubert, F., Krell, S.: Overlapping DDFV Schwarz algorithms on non-matching grids. In: Domain Decomposition Methods in Science and Engineering XXVI. Springer (2021)
3. Lions, P.L.: On the Schwarz alternating method I. In: R. Glowinski, G.H. Golub, G.A. Meurant, J. Périaux (eds.) First International Symposium on Domain Decomposition Methods for Partial Differential Equations, pp. 1–42. SIAM, Philadelphia, PA (1988)
4. Schwarz, H.A.: Über einen Grenzübergang durch alternierendes Verfahren. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich **15**, 272–286 (1870)