On the Choice of Parameters in a Loosely-Coupled Dirichlet-Neumann Scheme for Fluid-Structure Interaction

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1 Introduction

The numerical solution of the interaction between an incompressible viscous fluid and an elastic solid is particularly challenging, especially in regimes with high added mass, i.e. when fluid and structure densities are comparable [6, 9]. Such condition arises naturally in biomedical applications [15].

Numerical strategies for solving the fluid-structure interaction (FSI) problem can be categorized into *monolithic* and *partitioned* approaches [13, 3]. While monolithic schemes may require large memory allocation, suitable preconditioners, and ad-hoc code implementation, partitioned schemes take advantage of modularity to treat better-conditioned subproblems and to exploit existing standalone fluid and structural codes [19]. Within partitioned approaches, one can further distinguish between strongly-coupled (SC) schemes [14, 1, 12, 7, 18], which employ sub-iterations at each time step to enforce the interface conditions, and *loosely-coupled* (LC) schemes [11, 4, 2, 5, 10], where the two sub-problems are solved just once per time step. However, the stability of LC schemes is in general strongly compromised in the presence of large added-mass [6, 1]. In this regard, several studies have proposed LC schemes where the introduction of suitable interface conditions and/or stabilization terms, possibly combined with operator-splitting techniques, ensure conditional stability of the numerical solution [11, 4, 2, 5, 10, 19]. Nonetheless, existing LC approaches typically rely on non-standard interface formulations and/or depend on an appropriate a priori choice of parameters, such as Robin or Nitsche coefficients, whose identification can be non-trivial and problematic for complex, real-world FSI problems.

In this context, we presented in [17] a new LC scheme based on standard Dirichlet and Neumann interface coupling conditions (*LC-DN-α*) which is stable in the pres-

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ence of high added mass, provided that the model parameter α is suitably chosen. In [17], we demonstrated the stability of the method in several scenarios. However, some significant behaviours of the LC-DN-α scheme were not yet investigated. To address these gaps, the present work completes the computational analysis of the scheme provided in [17] by presenting new numerical experiments aimed at: introducing an effective strategy for selecting stable values of α; investigating how the stability of the scheme depends on the spatial and temporal discretization parameters.

2 Problem settings

Referring to Figure 1, we consider an FSI problem where the coupling conditions on the continuity of the fluid and structure velocities and normal stresses are enforced at the fluid-structure interface Σ and where we set $\boldsymbol{n}_f = -\boldsymbol{n}_s$ the outwards normal of the fluid (Ω_f^t) and solid (Ω_s^0)[1] domains, respectively.

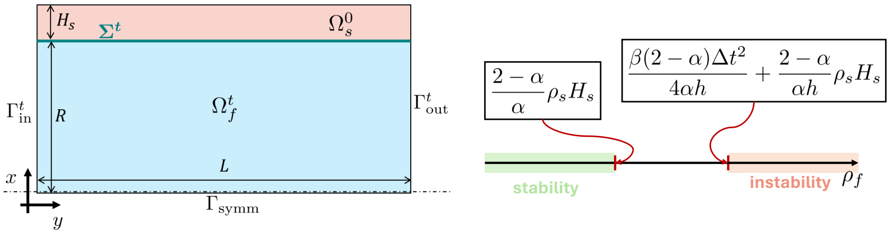

Fig. 1 Left: Fluid and structure domains. Right: Stability and instability regions in the space of ρ_f induced by conditions (3) and (2).

With these settings and given suitable initial conditions at $t = 0$ and boundary conditions on $\partial\Omega_f^t \setminus \Sigma^t$ and $\partial\Omega_s^t \setminus \Sigma^t$, the continuous FSI problem in the strong form reads as follows:

Find the fluid velocity $\boldsymbol{u}$*, the fluid pressure* p*, the structure displacement* $\boldsymbol{\eta}$*, and the structure interface velocity* $\boldsymbol{w}$ *satisfying* $\forall t \in (0, T]$*, with* T *the final time, the following problem*:

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[1] As usual, we consider here a Lagrangian representation of structure quantities. For the latter, we do not use any specific symbols

$$\begin{cases} \rho_f \left[\dfrac{\delta \boldsymbol{u}}{\delta t} + (\boldsymbol{u} - \boldsymbol{u}_{\text{ALE}}) \cdot \nabla \boldsymbol{u} \right] - \nabla \cdot \boldsymbol{T}_f(\boldsymbol{u}, p) = \boldsymbol{0} & \text{in } \Omega_f^t, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_f^t, \\ \boldsymbol{u} = \boldsymbol{w}, \qquad \boldsymbol{w} = \dfrac{\partial \boldsymbol{\eta}}{\partial t} & \text{on } \Sigma^t, \\ \boldsymbol{T}_f \cdot \boldsymbol{n}_f + \boldsymbol{T}_s \cdot \boldsymbol{n}_s = \boldsymbol{0} & \text{on } \Sigma^t, \\ \rho_s \dfrac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \nabla \cdot \boldsymbol{T}_s(\boldsymbol{\eta}) + \beta \boldsymbol{\eta} = \boldsymbol{0} & \text{in } \Omega_s^0, \end{cases} \tag{1}$$

where $\frac{\delta \boldsymbol{v}}{\delta t} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{u}_{\text{ALE}} \cdot \nabla)\boldsymbol{v}$ is the Arbitrary-Lagrangian Eulerian (ALE) time derivative with $\boldsymbol{u}_{\text{ALE}}$ the velocity of the fluid domain, obtained by harmonically extending the interface velocity $\boldsymbol{w}$ inside the fluid domain [8]; ρ_f and ρ_s represent the fluid and structure densities; $\boldsymbol{T}_f$ and $\boldsymbol{T}_s$ are the fluid and the structure Cauchy stress tensors defined as $\boldsymbol{T}_f = -p\boldsymbol{I} + \mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ and $\boldsymbol{T}_s = c(\nabla \boldsymbol{\eta} + (\nabla \boldsymbol{\eta})^T) + \lambda(\nabla \cdot \boldsymbol{\eta})\boldsymbol{I}$, with μ the dynamic fluid viscosity, c and λ the Lamé's constants. In the 2D case, the structure equation is also equipped by a reaction term ($\beta \neq 0$) which surrogates the circumferential elastic forces [1].

Regarding the numerical approximation of the FSI problem (1), we employ a Finite Elements discretization in space (with mesh parameter h) combined with a Finite Difference scheme in time (with parameter Δt) and a suitable linearization of the fluid convective term. At each time step, this setting yields the monolithic linear system $A\mathbf{X} = \mathbf{b}$ (current temporal index $n+1$ is understood), with A non-symmetric and definite positive[2].

3 The LC-DN-α scheme

In this section, we present the loosely-coupled scheme LC-DN-α proposed in [17]. This is obtained by performing just one iteration of the preconditioned Richardson method $P\mathbf{X}^{(k+1)} = P\mathbf{X}^{(k)} + \alpha \left(\mathbf{b}^{(k+1)} - A\mathbf{X}^{(k)}\right)$ (hence taking $k = 0$), where P is the standard Dirichlet-Neumann/Gauss-Seidel preconditioner [1] and α is a parameter that is not restricted to the value 1 as in the standard DN scheme. Consistency of the scheme is obtained by adding suitable terms [17].

The resulting LC-DN-α scheme is detailed in Algorithm 1. We refer the interested reader to [17] for its complete derivation.

3.1 Stability of the LC-DN-α scheme

For the study of the stability properties of Algorithm 1, we rely on the same benchmark problem described in [6, 10, 17] and we consider BDF1 schemes for the time

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[2] We changed the sign of the incompressibility equation to obtain definite positiveness [16]

Algorithm 1 LC-DN-α scheme

Given suitable initial and boundary conditions, find $\forall n = 0, 1, \dots, T/\Delta t$, $\boldsymbol{u}^{n+1}$, p^{n+1}, $\boldsymbol{\eta}^{n+1}$, *and* $\boldsymbol{w}^{n+1}$ *by solving in sequence* ($n+1$ *understood*):

1. Fluid problem equipped with a Dirichlet boundary condition at the interface Σ:

$$\begin{cases} \rho_f\left(\frac{\boldsymbol{u}}{\Delta t} + (\boldsymbol{u}^n - \boldsymbol{u}^n_{ALE})\cdot\nabla\boldsymbol{u}\right) - \nabla\cdot\boldsymbol{T}_f = \rho_f\frac{\boldsymbol{u}^n}{\Delta t} + & \\ \qquad + (1-\alpha)\left[\rho_f\left(\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t} + (\boldsymbol{u}^n - \boldsymbol{u}^n_{ALE})\cdot\nabla\boldsymbol{u}^n\right) - \nabla\cdot\boldsymbol{T}^n_f\right] & \text{in } \Omega_f, \\ \nabla\cdot\boldsymbol{u} = (1-\alpha)\nabla\cdot\boldsymbol{u}^n & \text{in } \Omega_f, \\ \boldsymbol{u} = (1-\alpha)\boldsymbol{u}^n + \alpha\boldsymbol{w}^n & \text{on } \Sigma; \end{cases}$$

2. Structure problem equipped with a Neumann boundary condition at the interface Σ:

$$\begin{cases} \rho_s\frac{\boldsymbol{\eta}}{\Delta t^2} - \nabla\cdot\boldsymbol{T}_s + \beta\boldsymbol{\eta} = \rho_s\frac{2\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}}{\Delta t^2} + & \\ \qquad + (1-\alpha)\left(\rho_s\frac{\boldsymbol{\eta}^n - 2\boldsymbol{\eta}^{n-1} + \boldsymbol{\eta}^{n-2}}{\Delta t^2} - \nabla\cdot\boldsymbol{T}^n_s\right) & \text{in } \Omega^0_s, \\ \boldsymbol{T}_s\cdot\boldsymbol{n}_s = -\boldsymbol{T}_f\cdot\boldsymbol{n}_f + (1-\alpha)\left(\boldsymbol{T}^n_s\cdot\boldsymbol{n}_s + \boldsymbol{T}^n_f\cdot\boldsymbol{n}_f\right) & \text{on } \Sigma^0; \end{cases}$$

3. Compute $\boldsymbol{w} = \left.\frac{\boldsymbol{\eta} - \boldsymbol{\eta}^n}{\Delta t}\right|_{\Sigma^0}$.

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discretization. In what follows we report two stability results related to the proposed scheme, see [17] for the proofs.

Proposition 1 *Assume* $0 < \alpha \leq 1$. *If*

$$\rho_f\mu_{min} > \frac{\beta(2-\alpha)\Delta t^2}{4\alpha} + \frac{2-\alpha}{\alpha}\rho_s H_s, \tag{2}$$

then the LC-DN-α scheme is unconditionally unstable.

Here H_s represents the structure's thickness and μ_{min} is the minimum eigenvalue of the added mass operator, whose behaviour for small h is $\mu_{min} \sim h$ [6, 10].

Proposition 2 *If*

$$\rho_f\mu_{max} < \frac{2-\alpha}{\alpha}\rho_s H_s, \tag{3}$$

then the LC-DN-α scheme is unconditionally absolute stable.

Here μ_{max} is the maximum eigenvalue of the added mass operator, whose behaviour for small h is $\mu_{max} \sim h^0$ [6, 10]. Notice that values $\alpha > 1$ are not considered since they are expected to lead to unstable solutions, see Proposition 3 in [17].

3.2 Discussion on the stability and instability conditions

First of all, we observe that conditions (2) and (3) characterize distinct subsets in the space of ρ_f, as illustrated in Figure 1-right. In what follows, we go deeper into the discussion of the dependence of the stability behaviour of the LC-DN-α scheme on the space and time discretization parameters. Moreover, we provide a strategy for an effective guess for α.

1. **Choice of the parameter α:** It is well known that the optimal value, $\alpha_{opt} = \frac{2}{\lambda_{min}+\lambda_{max}}$ (with λ_{min} and λ_{max} the minimum and maximum eigenvalues of the system matrix), predicted by the theory of the Richardson method is not proven to be optimal for matrices that are non-symmetric [16] as in our case. Nevertheless, since the LC-DN-α scheme is inspired by the Richardson method, we considered the optimal value as a reasonable initial guess for identifying stable parameters α.
2. **Dependence on h:** From Proposition 1 it follows that condition (2) depends on the space discretization parameter h. In particular, mesh refinement appears to shift the instability threshold for ρ_f towards higher values, see Figure 1-right.
3. **Dependence on Δt:** From (2) we observe that reducing Δt the right hand side contribution quadratically reduces too, hence shifting the instability threshold for ρ_f to lower values, see Figure 1-right. By isolating the contribution of the time step to the left-hand side we obtain:

$$(2-\alpha)\Delta t^2 < \frac{-8\rho_s H_s + 4\alpha(\rho_f \mu_{min} + \rho_s H_s)}{\beta},$$

   which, combined with the assumption of Proposition 1 ($0 < \alpha \leq 1$), yields the following result:

$$\text{if } \frac{2\rho_s H_s}{\rho_s H_s + \rho_f \mu_{min}} < \alpha < 1, \text{ the scheme is unstable for}$$
$$\Delta t < \sqrt{\frac{4\alpha(\rho_s H_s + \rho_f \mu_{min}) - 8\rho_s H_s}{\beta(2-\alpha)}}.$$

   Remark 1 Notice that, although the previous condition on Δt may appear counterintuitive, suggesting instability for sufficiently small time steps, the LC-DN-α scheme remains consistent as $\Delta t \longrightarrow 0$, provided that α is small enough as appears from condition (2).

4 Numerical experiments

In this section, we present some numerical experiments performed by applying the LC-DN-α scheme to problem (1). Unless otherwise specified and referring to Figure

1-left, we consider a rectangular structure and fluid domains with $L = 6cm$, $R = 0.5cm$, and $H_s = 0.1cm$, $\mu = 0.035poise$, $\rho_f = 1.1g/cm^3$, $\rho_s = 1g/cm^3$, $c = 5 \times 10^5 dyne/cm^2$, $\lambda = 7.5 \times 10^6 dyne/cm^2$, $\beta = 5.7 \times 10^6 dyne/cm^4$. We impose the inlet pressure profile $p_{in}(t) = 2 \times 10^4 \left(1 - \cos\left(\frac{\pi t}{0.05}\right)\right) dyne/cm^2$ for $t \leq 0.1s$, and $p_{in}(t) = 0\, dyne/cm^2$ for $0.1s < t < T$. All these settings are inspired by hemodynamic applications.

The simulations are carried out using a 2D Finite Elements code written in Matlab. For the numerical settings, we employ the BDF1 schemes for the fluid and structure time discretizations. The convective term of the Navier-Stokes equations is linearized by means of a first order extrapolation. In space we use $\mathbb{P}_1 - iso\mathbb{P}_2/\mathbb{P}_1$ Finite Elements for the fluid and $\mathbb{P}_1$ for the structure, with a reference discretization step $h = 0.2cm$. With this mesh, each time step takes about 1 minute to run. In what follows we present three numerical tests. In each case we report as a result the pressure in time at the middle section of the domain.

4.1 Test I: choice of the parameter α

In the first test case, we study the stability properties of LC-DN-α scheme by varying the value of α with $\Delta t = 5 \times 10^{-4}s$. In Figure 2-A, we report an example of using the optimal value of the acceleration parameter predicted by the theory of the Richardson method, see point 1 in Section 3.2. In particular, we compute $\alpha_{opt} = \frac{2}{Re(\lambda_{min})+Re(\lambda_{max})}$ with λ_{min} and λ_{max} the minimum and maximum eigenvalues of the preconditioned monolithic FSI matrix. We note that this choice of α ensures stability of the solution (pink dashed line); even a slight increase in α results in an unstable behaviour (yellow line), in accordance with condition (2). Moreover, for α_{opt} the solution is more accurate than for lower values of α (green dashed line and light blue line). Notice that the computation of α_{opt} does not affect the overall computational times since it is performed off-line, being α_{opt} constant in time. These outcomes suggest that our proposal about α_{opt} is very effective.

4.2 Test II: dependence on h

To analyse how the space discretization influences the stability of the scheme, we neglect in (1) the fluid convective term (Stokes problem), in order to avoid numerical issues due to elevated Reynolds numbers, which could make more confusing the analysis of the results [20]. We set $R = 1cm$, $\rho_s/\rho_f = 0.05$, $\Delta t = 2.5 \times 10^{-4}s$, and $\alpha = 0.0064$. In Figure 2-B, we report the numerical results obtained for different values of h. These results confirm the theoretical findings (see point 2 in Section 3.2), i.e. that for too coarse meshes the LC-DN-α scheme is unstable.

4.3 Test III: dependence on Δt

In the last test, for $\alpha = 0.05$, we change the value of Δt, obtaining that the solution becomes unstable if the Δt is too small, see Figure 2-C. This confirms the theoretical findings presented in point 3, Section 3.2. However, as highlighted in Remark 1, we can recover stability by suitably decreasing the value of α, see Figure 2-D.

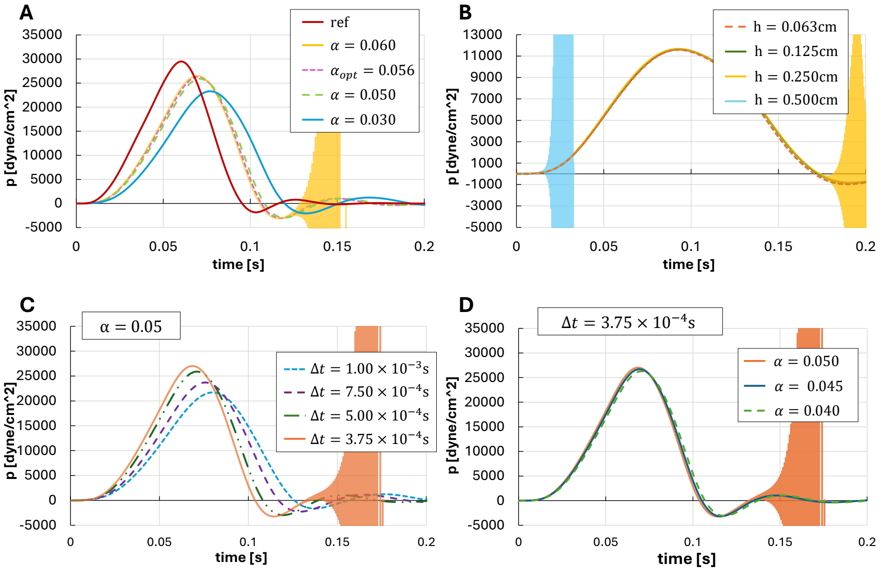

Fig. 2 Fluid mean pressure over the cross-section at $z = 3\,cm$. A: different values of α for $\Delta t = 5 \times 10^{-4}$; red line: reference monolithic solution (Test I). B: different values of h for $R = 1cm$, $\rho_s/\rho_f = 0.05$, $\Delta t = 2.5 \times 10^{-4}s$, and $\alpha = 0.0064$ (Test II). C: different values of Δt for $\alpha = 0.05$ (Test III). D: different values of α for $\Delta t = 3.75 \times 10^{-4}s$ (Test III).

All these results showed that the proposed LC scheme for the first time is able to lead to stable results in hemodynamics by using standard Dirichlet and Neumann interface conditions (instead of Robin or Nitsche ones as in existing schemes).

We conclude by noticing that in this work we have focused on 2D benchmark simulations. It will be mandatory for future studies to address the validity of our theoretical findings to more complex 3D geometries and nonlinear materials.

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