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POLITECNICO DI MILANO

Heterogeneous DD - An overview

Heterogeneous DD: Motivations

1. Induced by Mathematical Models

- Advection diffusion / advection
- Exterior aerodynamics:
  Euler/compressible NS,
  Incompressible NS/potential
- Maxwell in heterogeneous media …..

2. Geometrical Multiscale (ex. 3D-1D, 3D-0D)

- Blood flow: modeling the whole circulatory system
- Environment (river – wide water bodies)
- Semiconductor devices (electronic circuits) ….

3. Multiphysics

- Fluid structure interaction (FSI)
  Aerodynamics
  Dam / reservoirs
  Blood / vessel wall
- Surface / subsurface flows …..

Mathematical Approaches

1. Non-overlap / overlap

2. Treatment of interfaces

- Interface matching (physically driven: conservation principles, equilibration of solution and “stresses”)

- Asymptotics on critical physical parameter (e.g., Reynolds, magnetic/electric conductivity,…)

- Asymptotics on variational solution

- Control (distributed vs boundaries), virtual control

HDD

INDUCED BY MATHEMATICAL MODELS

Model hierarchy in CFD

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho\vec{\mathbf{u}}\right) = 0$$
$$\frac{\partial\left(\rho\vec{\mathbf{u}}\right)}{\partial t} + \mathrm{div}\left(\rho\vec{\mathbf{u}} \otimes \vec{\mathbf{u}}\right) = -\vec{\nabla}p + \boxed{\cancel{\mathrm{div}\,(\mathcal{T})}} + \rho\vec{\mathbf{f}}$$
$+$ energy equation

$e = e(\rho, T), \quad p = p(\rho, T)$

Compressible Euler equations

$$\boxed{\cancel{\frac{\partial \rho}{\partial t}}} + \mathrm{div}\left(\rho\vec{\mathbf{u}}\right) = 0$$
$$\frac{\partial\left(\rho\vec{\mathbf{u}}\right)}{\partial t} + \mathrm{div}\left(\rho\vec{\mathbf{u}} \otimes \vec{\mathbf{u}}\right) = -\vec{\nabla}p + \mathrm{div}\,(\mathcal{T}) + \rho\vec{\mathbf{f}}$$
$$\boxed{\cancel{+ \text{ energy equation} \quad e = e(\rho, T), \quad p = p(\rho, T)}}$$

Incompressible Navier-Stokes equations

Neglecting viscous effects

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho\vec{\mathbf{u}}\right) = 0$$
$$\frac{\partial\left(\rho\vec{\mathbf{u}}\right)}{\partial t} + \mathrm{div}\left(\rho\vec{\mathbf{u}} \otimes \vec{\mathbf{u}}\right) = -\vec{\nabla}p + \mathrm{div}\,(\mathcal{T}) + \rho\vec{\mathbf{f}}$$
$+$ energy equation

$e = e(\rho, T), \quad p = p(\rho, T)$

Compressible Navier-Stokes equations

Neglecting compressibility effects

Considering the fluid as irrotational

Considering the fluid as irrotational

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{c^2}\nabla\phi\cdot\left[\left(\nabla\phi\cdot\nabla\right)\nabla\phi\right] + \frac{2}{c^2}\nabla\phi\cdot\frac{\partial\nabla\phi}{\partial t} - \Delta\phi = 0$$

Full potential equation

Neglecting nonlinear terms and time

$$\boxed{\cancel{\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{c^2}\nabla\phi\cdot\left[\left(\nabla\phi\cdot\nabla\right)\nabla\phi\right] + \frac{2}{c^2}\nabla\phi\cdot\frac{\partial\nabla\phi}{\partial t}}} - \Delta\phi = 0$$

Laplace equation

HDD (Heterogeneous Domain Decomposition)

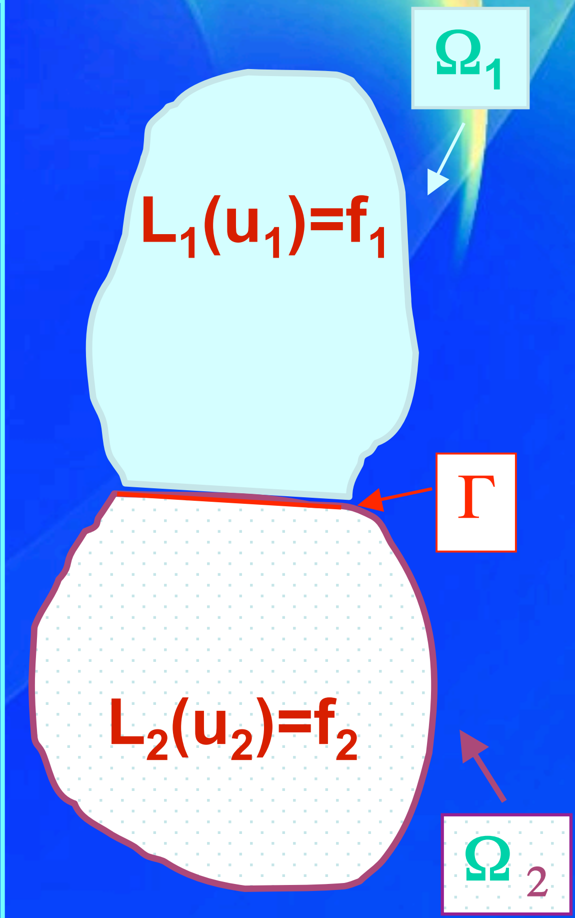
ON NON-OVERLAPPING

SUBDOMAINS

1. Abstract Setting

2. Advection-Diffusion

3. Navier-Stokes/Oseen



Heterogeneous DD (mathematically induced)

An abstract setting

Consider two *different kind* of boundary value problems within two disjoint sub-regions

$$\textit{find}\ u_1 : \ L_1 u_1 = f_1 \ in \ \Omega_1$$

$$\textit{find}\ u_2 : \ L_2 u_2 = f_2 \ in \ \Omega_2$$

The unknowns u_1 and u_2 should satisfy proper matching conditions, say

$$\Phi(u_1) = \Phi(u_2) \ on \ \Gamma_1$$

$$\Psi(u_1) = \Psi(u_2) \ on \ \Gamma_2$$

with Γ_1 and Γ_2 suitable subsets of the interface Γ

HDD for Advection-Diffusion

$$
\begin{aligned}
L_2 u = f \quad & \text{in } \Omega \subset \mathbb{R}^d, \;\; d = 1,2 \\
u = g \quad & \text{on } \partial\Omega
\end{aligned}
$$

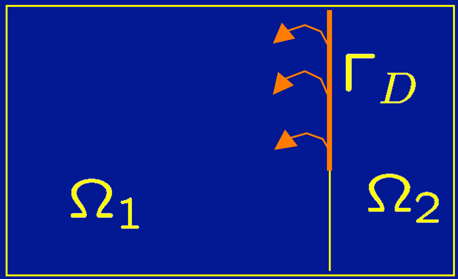
$$L_2 u := -\nu \Delta u + \mathrm{div}(\mathbf{b} u) + b_0 = -\nu \Delta u + L_1 u$$

The **heterogeneous** coupling:

$$L_1 u_1 = f \quad \text{in } \Omega_1, \qquad L_2 u_2 = f \quad \text{in } \Omega_2,$$

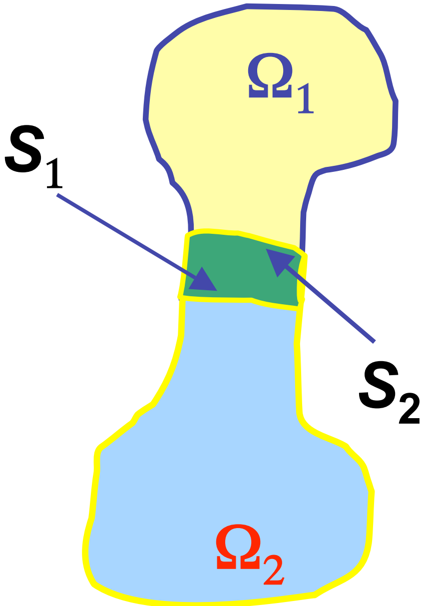
$$u_1 = u_2 \quad \text{on } \Gamma_D,$$

$$\phi_1 = \phi_2 \quad \text{on } \Gamma.$$



where ϕ are the normal flux

(Gastaldi-Q., DD3 (1989), Q.- Valli (1999))



MODEL REDUCTION by

VIRTUAL CONTROL on OVERLAPPING DOMAINS

(distributed or boundary control)

(R.Glowinski, O.Pironneau, JLLions)

Boundary control

The original problem:

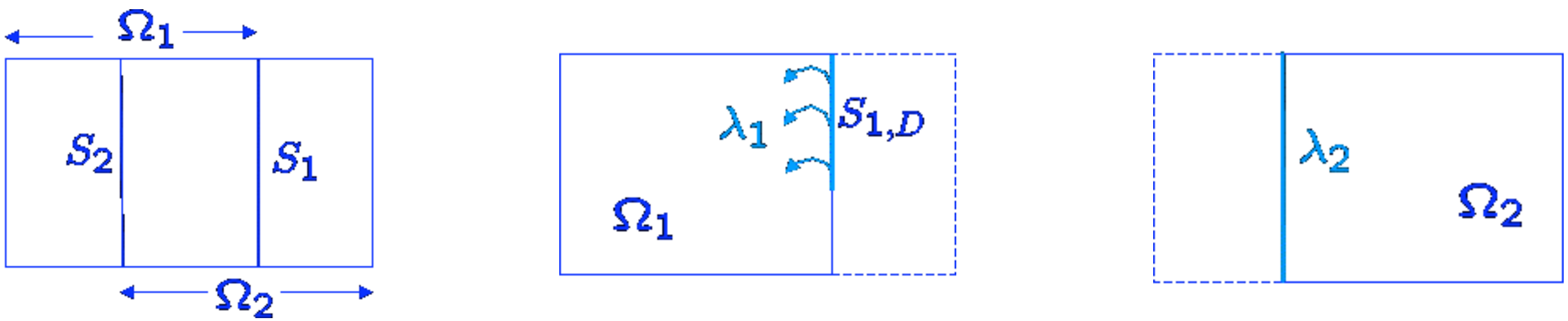
$$\boxed{\begin{array}{ll} L_2 u = f & \text{in } \Omega \subset \mathbb{R}^d,\ d = 1,2 \\ u = g & \text{on } \partial\Omega \end{array}}$$

with $L_2 u := -\nu \Delta u + \mathrm{div}(\mathbf{b}u) + b_0 = -\nu \Delta u + L_1 u$

The **heterogeneous** coupling:

$$\begin{cases} L_1 u = f & \text{in } \Omega_1 \\ u_1 = \lambda_1 & \text{on } S_{1,D}, \end{cases} \qquad \begin{cases} L_2 u = f & \text{in } \Omega_2 \\ u_2 = \lambda_2 & \text{on } S_2, \end{cases}$$

$\lambda_1, \quad \lambda_2$
Virtual controls



L^2- control: solve $\boxed{\inf_{\lambda_1,\lambda_2} J(\lambda_1,\lambda_2)}$

with $J(\lambda_1,\lambda_2) := \frac{1}{2}\int_{\Omega_1\cap\Omega_2} (u_1(\lambda_1) - u_2(\lambda_2))^2 d\Omega$

Lemma. If all data are smooth enough and if

$\mathbf{b}\cdot\mathbf{n} \neq 0$ on $\Sigma \subseteq \partial(\overline{\Omega_1\cap\Omega_2})\cap\Gamma$, then
$\inf_{\lambda_1,\lambda_2} J(\lambda_1,\lambda_2)$ admits a solution.

Theorem. If we set $\phi(\nu) = \inf_{\lambda_1,\lambda_2} J(\lambda_1,\lambda_2)$ and

if we let $\nu \to 0$, all other data being fixed, then

$$\phi(\nu) \to 0 \text{ as } \nu \to 0.$$

(Gervasio, J.-L. Lions, Q. , 2001)

REDUCING COMPLEXITY

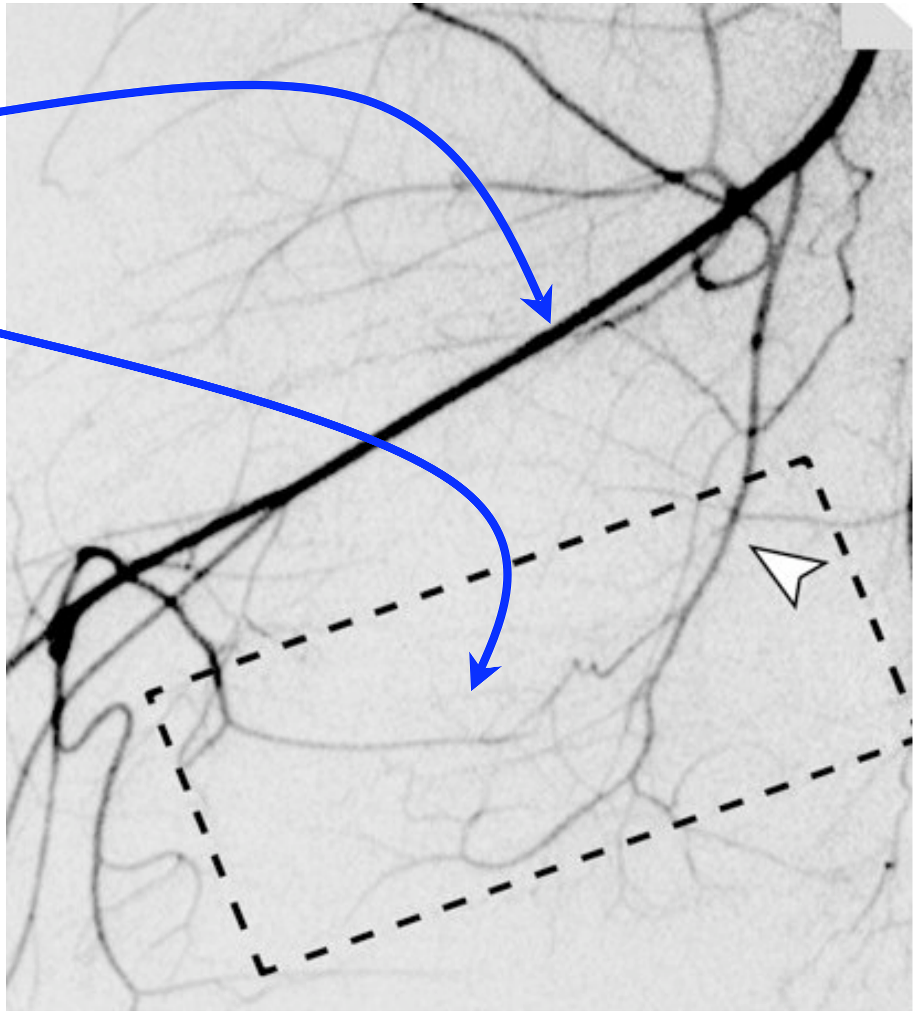
BY GEOMETRICAL MULTISCALE

Perfusion of tissues and vascular networks

The vascular structures of living tissues feature different scales (branching)

Medium size vessels:
1D structures

Small vessels (capillaries):
3D porous structures



Goal: numerical simulation of

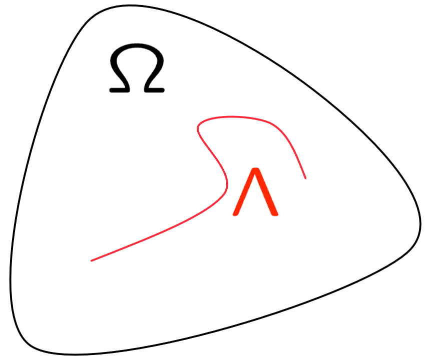
•Blood flow

•Metabolism, Transport

(C.D'Angelo)

The 1D-3D model problem

The simplest situation: *steady flow*.



$p_{\mathrm{t}} : \Omega \to \mathbb{R}$ blood pressure in the tissue (3D)

$p_{\mathrm{v}} : \Lambda \to \mathbb{R}$ blood pressure in the vessel (1D)

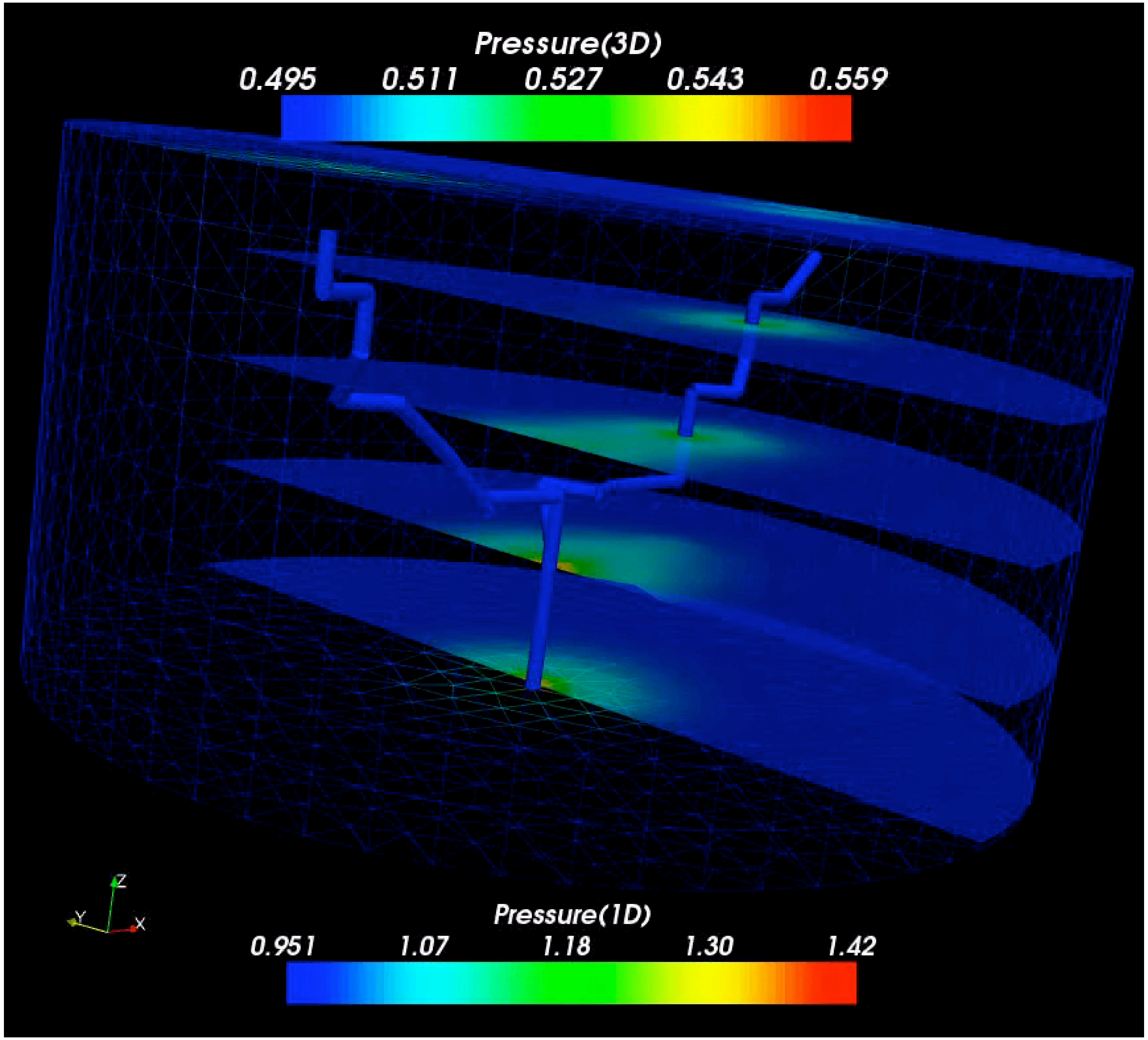
$\phi : \Lambda \to \mathbb{R}$ flow rate from the vessel to the tissue (exchange term)

$$\begin{cases} -\nabla \cdot (k_{\mathrm{t}} \nabla p_{\mathrm{t}}) - \phi \delta_{\Lambda} = 0 & \text{in } \Omega, \\ -\dfrac{\mathrm{d}}{\mathrm{d}s}\left(k_{\mathrm{v}} \dfrac{\mathrm{d}}{\mathrm{d}s} p_{\mathrm{v}}\right) + \phi = 0 & \text{in } \Lambda, \end{cases}$$

+ boundary conditions (3D: homogeneous Neumann, 1D: Dirichlet)

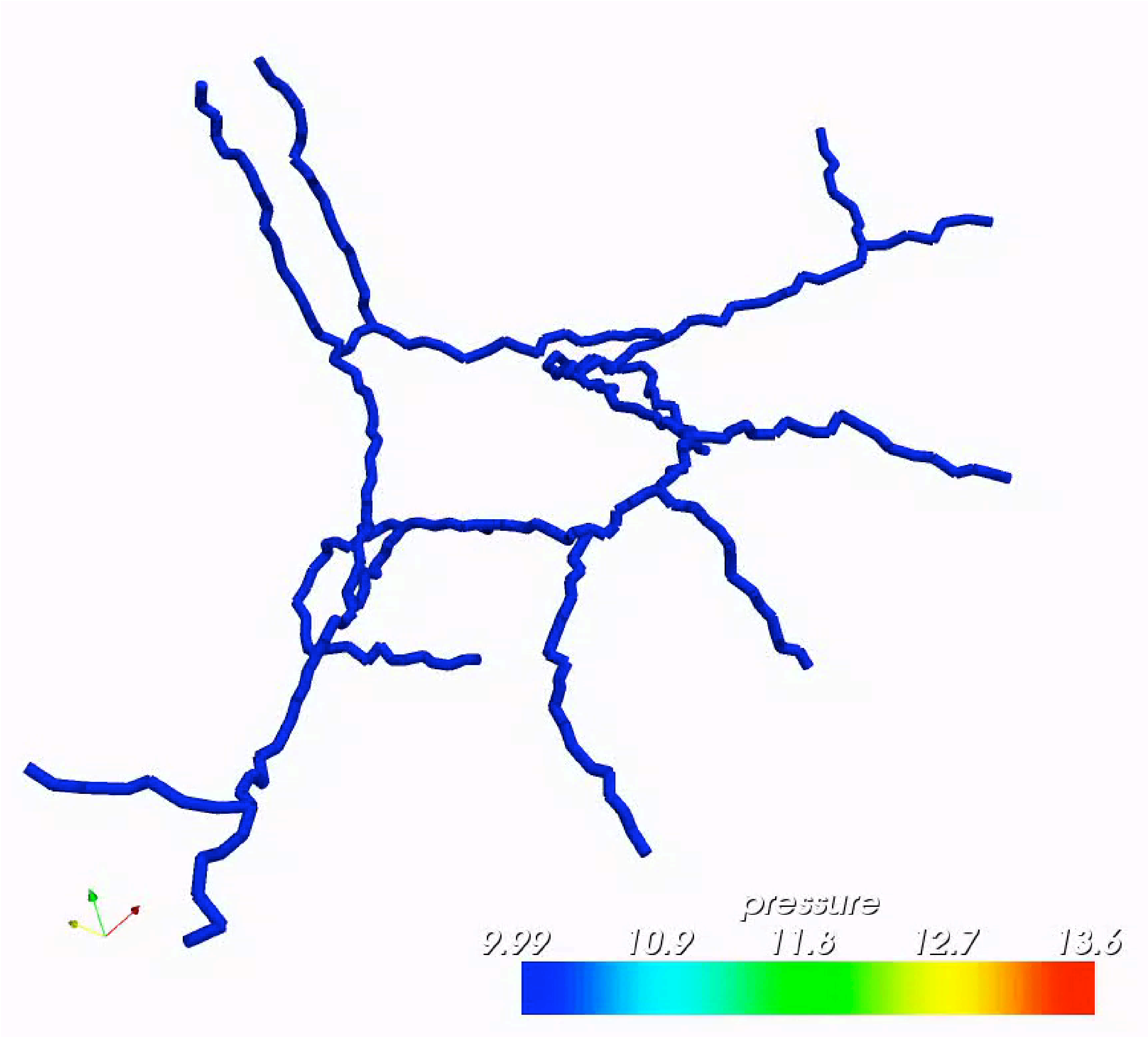
Notice the *measure term* in the 3D equation (Dirac measure concentrated on the vessel)

Flow in a vascularized tissue



Blood flow and oxygen transport (brain)

Circle of Willis



Oxygen transport and reaction in the tissue

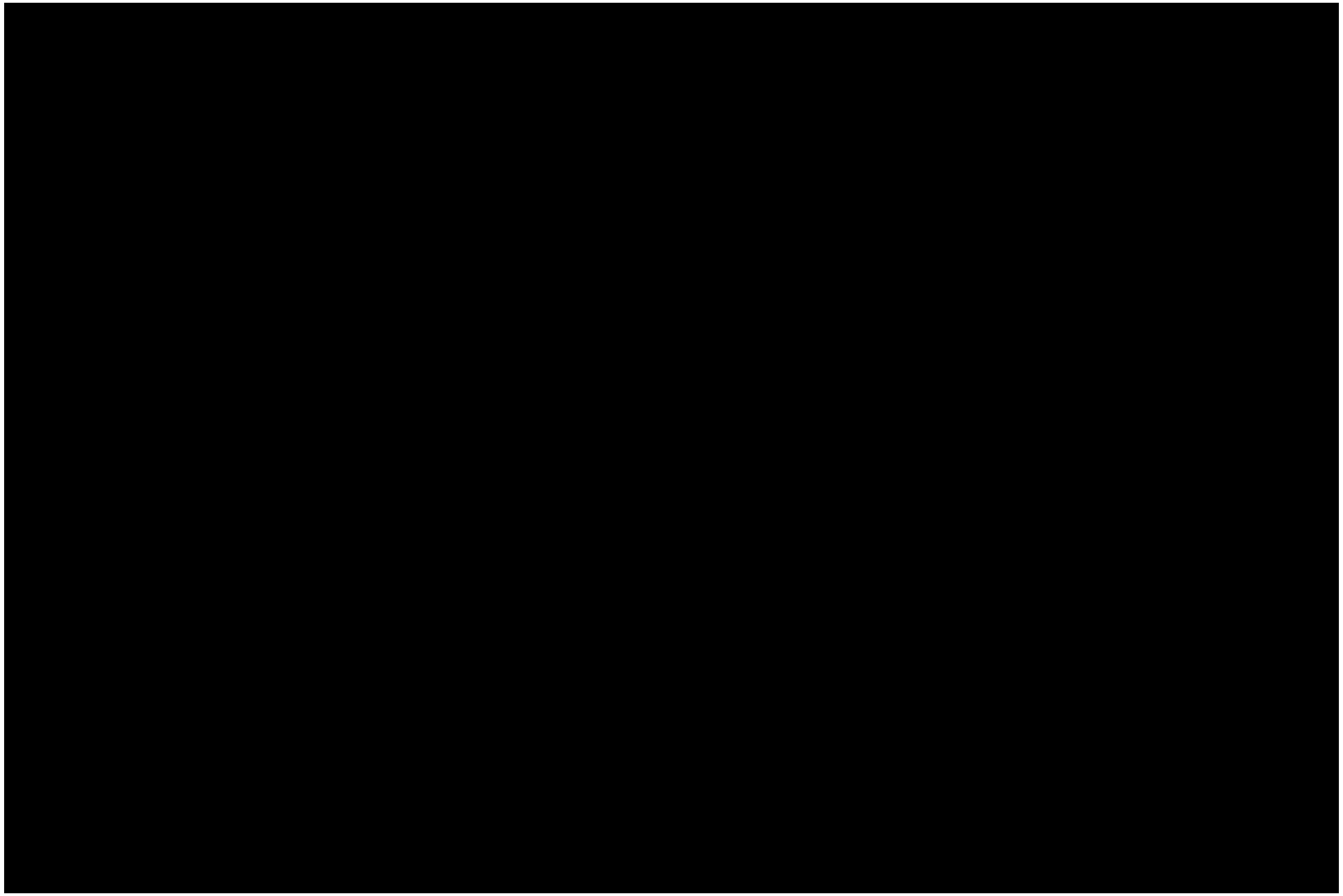
Blood pressure in the main vessels

Geometrical multiscale in circulatory system

Local (level1):
3D flow model

Global (level 2):
1D network of major arteries and veins

Global (level 3):
0D capillary network



3D-1D for the carotid: pressure pulse

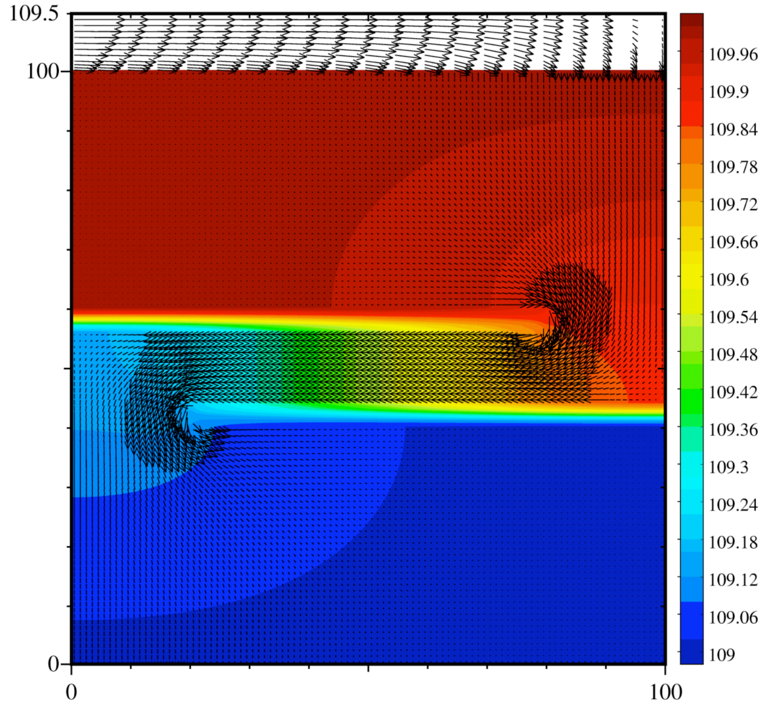
(A.Moura)

HDD

FOR MULTIPHYSICS

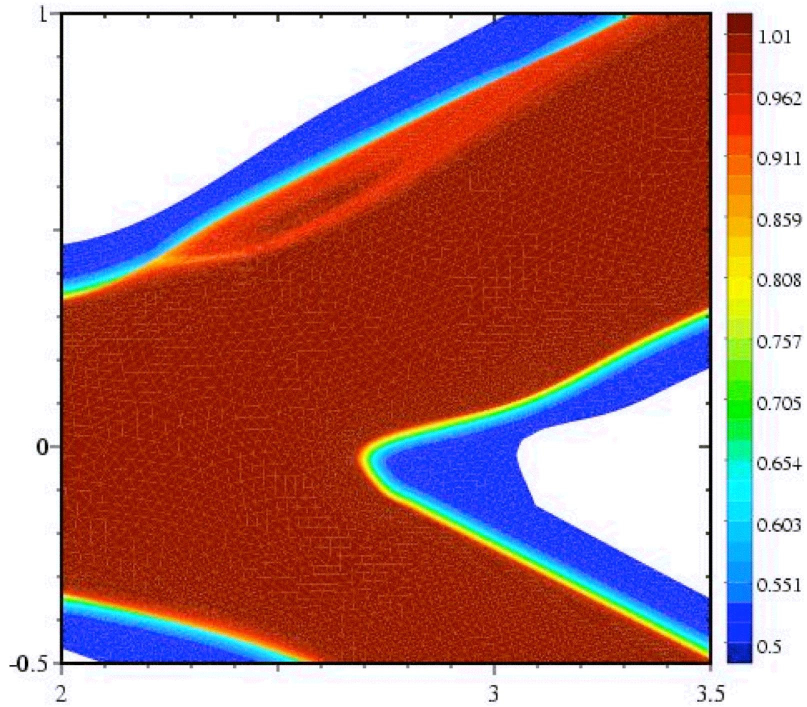
HDD FOR MULTIPHYSICS: Navier-Stokes/Darcy

Surface-Groundwater Flows



(E. Miglio)

Biochemical Transfer in Artery Walls



(P.Zunino)

Multiphysics in Sailing Yachts

Two-phase flow equations

Air Phase

$$\frac{\partial(\rho_a \boldsymbol{u}_a)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_a \boldsymbol{u}_a \otimes \boldsymbol{u}_a) - \boldsymbol{\nabla} \cdot \boldsymbol{T}_a(\boldsymbol{u}_a, p_a) = \rho_a \boldsymbol{g}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_a = 0$$

Interface Conditions

$$\boldsymbol{u}_a = \boldsymbol{u}_w \quad \text{on } \Gamma$$

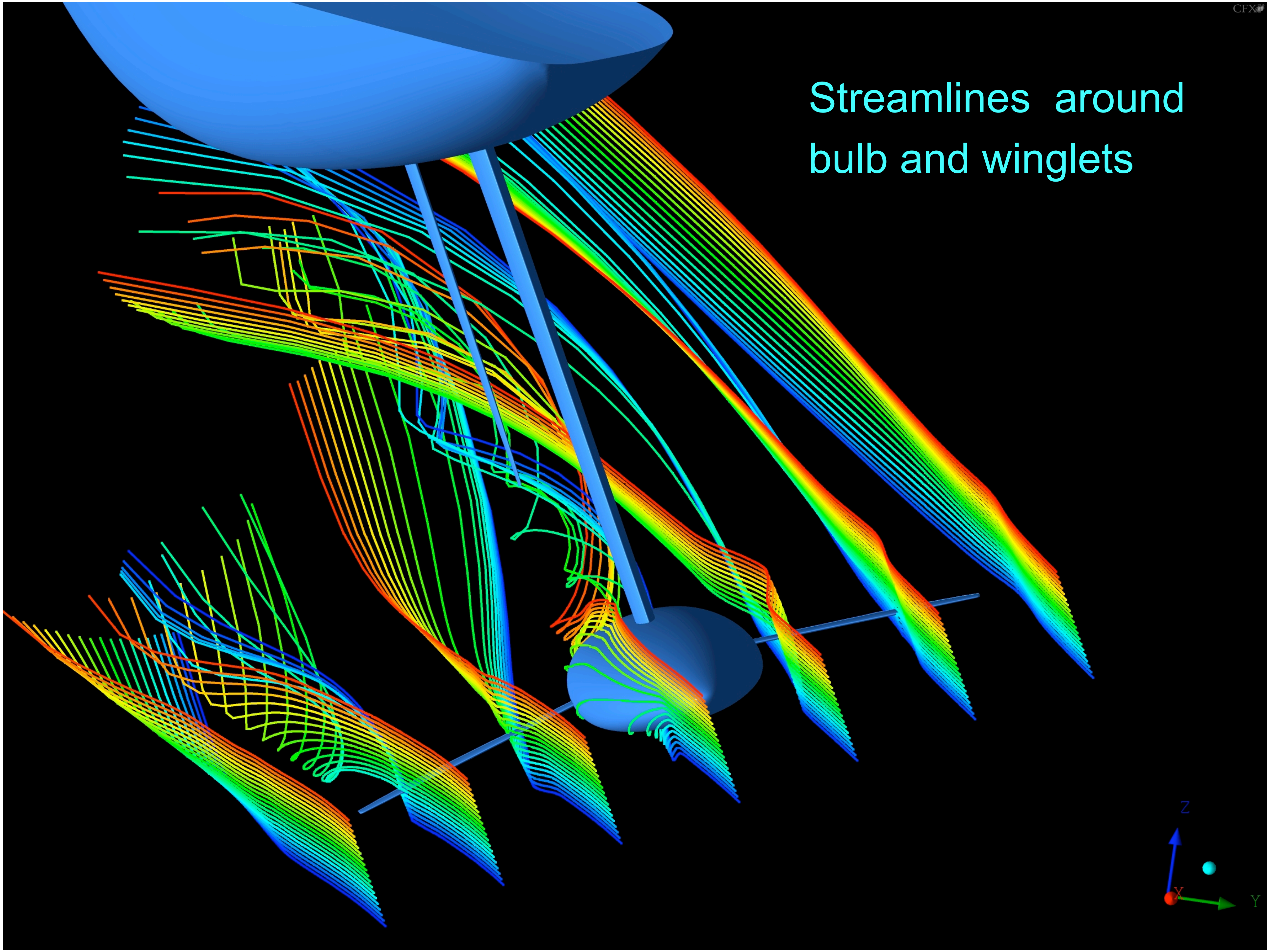
$$\boldsymbol{T}_a(\boldsymbol{u}_a, p_a) \cdot \boldsymbol{n} = \boldsymbol{T}_w(\boldsymbol{u}_w, p_w) \cdot \boldsymbol{n} + \kappa \sigma \boldsymbol{n} \quad \text{on } \Gamma$$

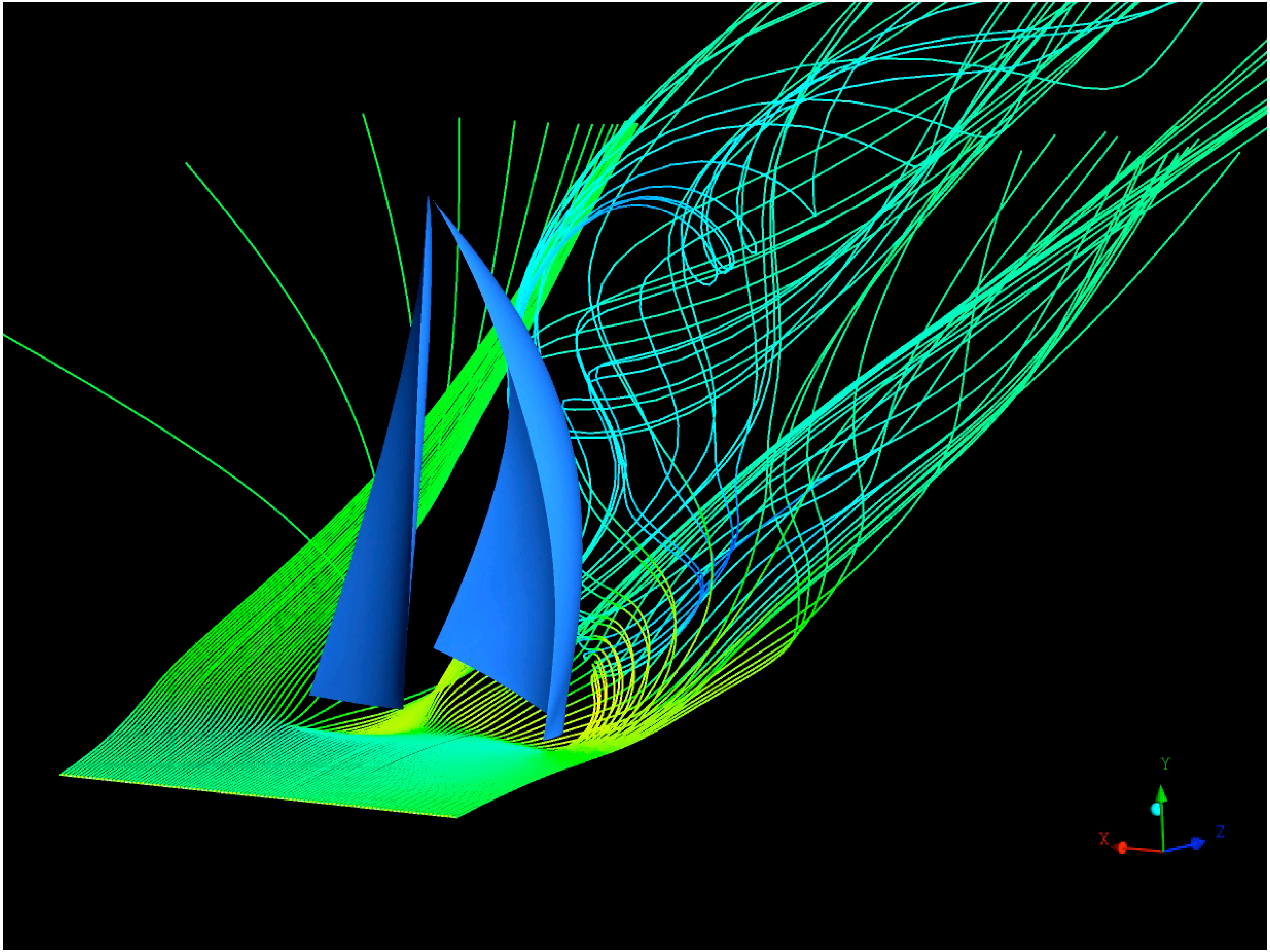
$$\frac{\partial(\rho_w \boldsymbol{u}_w)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_w \boldsymbol{u}_w \otimes \boldsymbol{u}_w) - \boldsymbol{\nabla} \cdot \boldsymbol{T}_w(\boldsymbol{u}_w, p_w) = \rho_w \boldsymbol{g}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_w = 0$$

Water Phase

Multiphysics in Sailing Yachts


Streamlines around
bulb and winglets
CFX
Z
X
Y


Y
X
Z

Multiphysics in the circulatory system

Fluid-vessel mechanical interaction

Blood-flow equations:

$$\frac{\partial \mathbf{u}}{\partial t}\Big|_{\hat{\mathbf{x}}} + \mathrm{div}\left(\mathbf{u} \otimes (\mathbf{u} - \mathbf{w}) - \frac{1}{\rho^f} \mathrm{T}(\mathbf{u}, p)\right) = 0 \text{ in } \Omega_f(t)$$

$$\mathrm{div}\ \mathbf{u} = 0 \text{ in } \Omega_f(t)$$

Vessel equation:

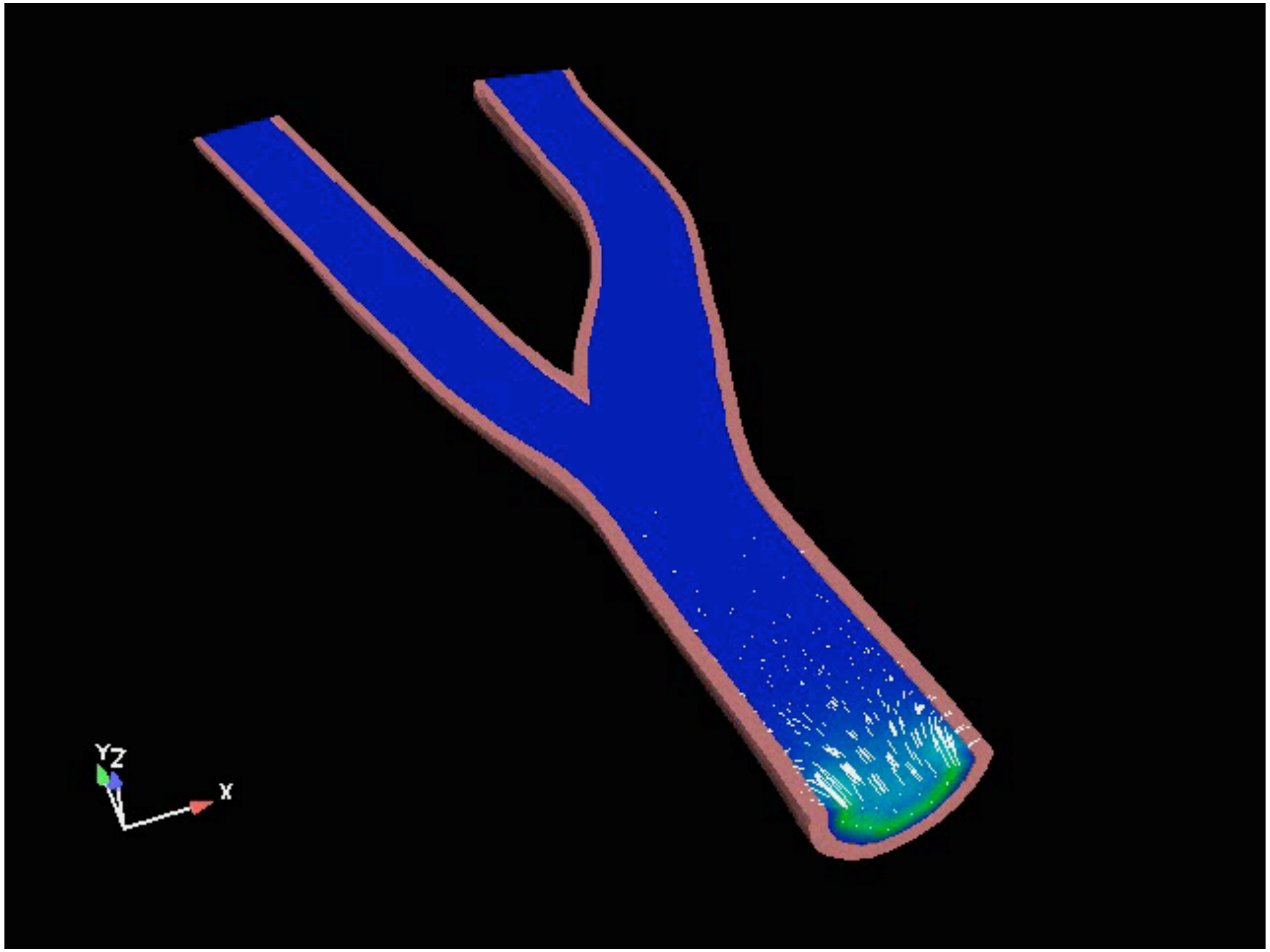
$$\frac{\partial^2 \eta}{\partial t^2} - \mathrm{Div}(\sigma(\eta)) = f(\eta) \text{ in } \Omega_s$$

Coupling equations:

$$\sigma(\eta) \cdot \mathbf{n} = \mathrm{T}(\mathbf{u}, p) \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\mathbf{u} = \frac{\partial \eta}{\partial t} \quad \text{on } \Gamma$$

Flowfield and vessel deformation



Solution Algorithms

1. Steady Problems

- Fixed point
- Substructuring based
- Steklov-Poincaré based
- Optimal "Fourier" algorithms *(Nataf, Halpern, Gander,Japhet)*
- CG + adjoint problem in control approaches

2. Unsteady Problems

- Monolithic
- Explicit
- Implicit + subiterations / DD
- Semi-implicit: differential / algebraic

Interface Problem:

Steklov-Poincare' equation

$$\boxed{SP_f(\lambda) + SP_s(\lambda) = 0}$$

Construction of the Steklov-Poincare' (**Dirichlet-to-Neumann**) maps SP_f and SP_s:

$$\boxed{\lambda \rightarrow Fl(\lambda, \mathbf{u}, p; \mathbf{v}) = 0 \rightarrow SP_f(\lambda) = \sigma_f(\mathbf{u}, p) \cdot \mathbf{n}_f}$$

$$\boxed{\lambda \rightarrow St(\eta, \lambda; \mathbf{v}_0^s) = 0 \rightarrow SP_s(\lambda) = \sigma_s(\eta) \cdot \mathbf{n}_s}$$

Note: here the St solver takes a displacement as bc on the interface.

Linear Steklov-Poincaré equation

Consider the problem

$$\text{find } \lambda \text{ in } X: \quad S\lambda = c$$

where $S: X$ *to* X' is a linear invertible continuous operator, such that

$$S = S_1 + S_2$$

If

- S_2 is continuous and coercive
- S_1 is continuous and non-negative,

then, for all λ^0 in X and for all $0<q<q^*$, the sequence

$$\lambda^{k+1} = \lambda^k + q\, S_2^{-1}(c - S\,\lambda^k)$$

converges in X to the solution λ.

(Q. and Valli (1999))

Nonlinear Steklov-Poincaré equation

Consider the problem

$$\text{find } \; l \text{ in } X: \quad S_1(l) + S_2 l = c$$

where S_1:*X à X'* is nonlinear and S_2:*X à X'* is linear.

If

- S is strongly monotone in X
- S_2 is continuous and coercive
- S_1 is Lipschitz continuous,

then, the solution is unique and for all l^0 in X and for all 0<q<q*, the sequence

$$l^{k+1} = l^k + q\, S_2^{-1} (c - S\, l^k)$$

converges in X to the solution l.

(H.Berninger (2007))

(proof based on Banach's fixed point Theorem)

Newton / fixed-point techniques

Consider the nonlinear problem

$$\text{find} \quad l \text{ in } X: \quad S(l) = 0$$

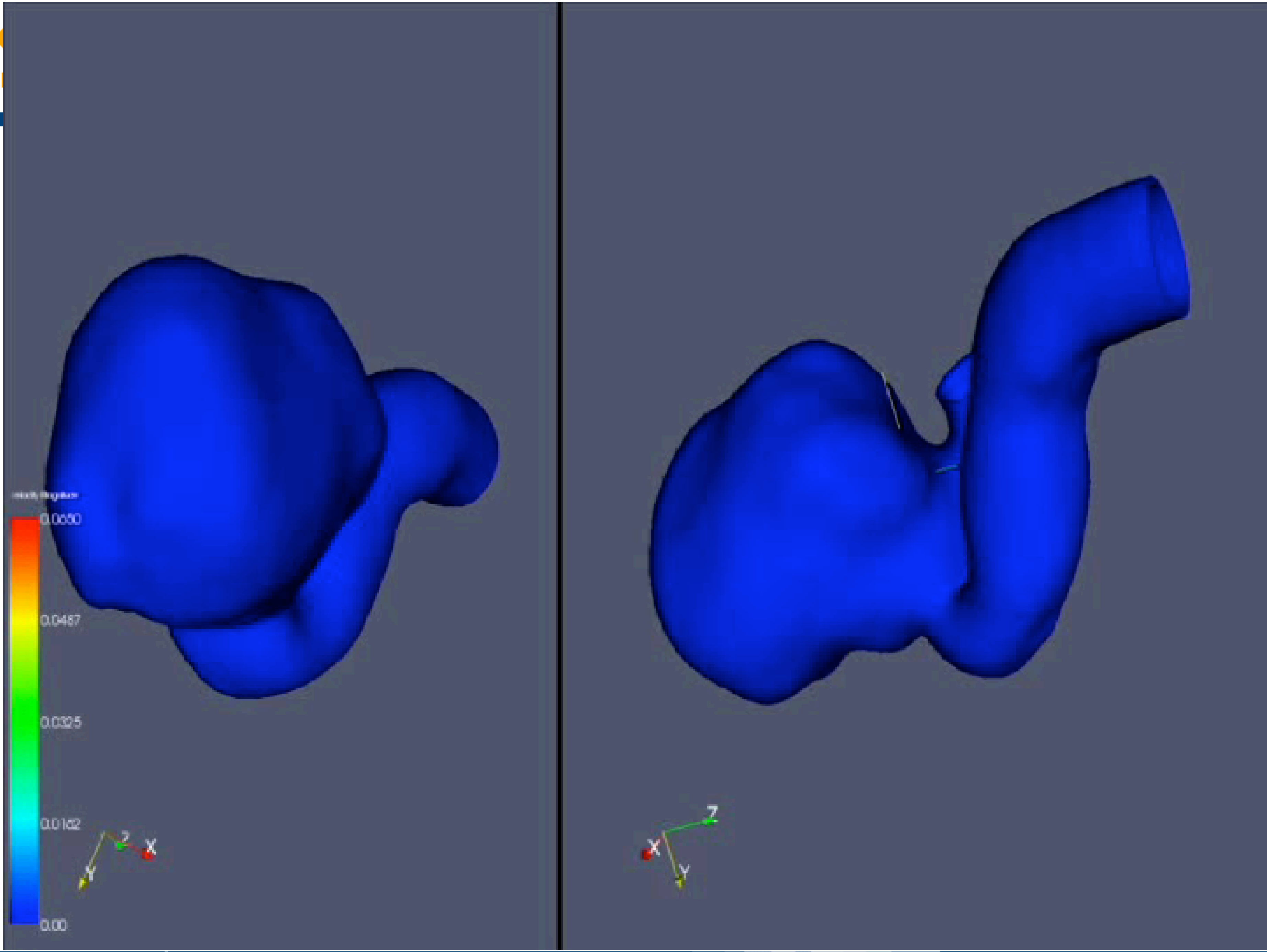
where *S:X à Y* is a nonlinear operator.
Let l^0 be "close enough" to the solution l in *X*.
If

- *S* has continuous second derivative in a suitable closed ball X_0 centered in l^0
- the linear operator $S'(l^0)^{-1}$ exists and is continuous
- $\| S'(l^0)^{-1} S(l^0)^{-1} \| < C_1$
- $\| S'(x)^{-1} S''(x) \| < C_2$ in X_0
- $C_1 C_2 < 1/2$,

then, the Newton method

$$l^{k+1} = l^k - S'(l^k)^{-1} S(l^k)$$

converges in X_0 to the solution l.
(Kantorovic (1948))



(S.Deparis-G.Fourestey)