FETI methods

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return on innovation

Outline

- FETI: main features
- Rigid body projection

- FETI-2: general methodology for coarse grid preconditioners
- Avoiding dense projector: FETI-DP
- Mixed equations: YADP-FETI

- Time harmonic equations: FETI-H

- Mixed interface conditions: FETI-2LM
- Algebraic approximate optimal interface conditions

- Conclusion

FETI: main features

- First method based on Lagrange multipliers
  - ⇒ discontinuous solution
  - ⇒ dual interface variables
- Easy interface management (uncoupling)
- Alternate cheap local preconditioner
- Imbedded coarse grid preconditioner

FETI : continuous model for elasticity

- Global problem

[illegible]

[illegible]

[illegible]

- Local problem

[illegible]

[illegible]

[illegible]

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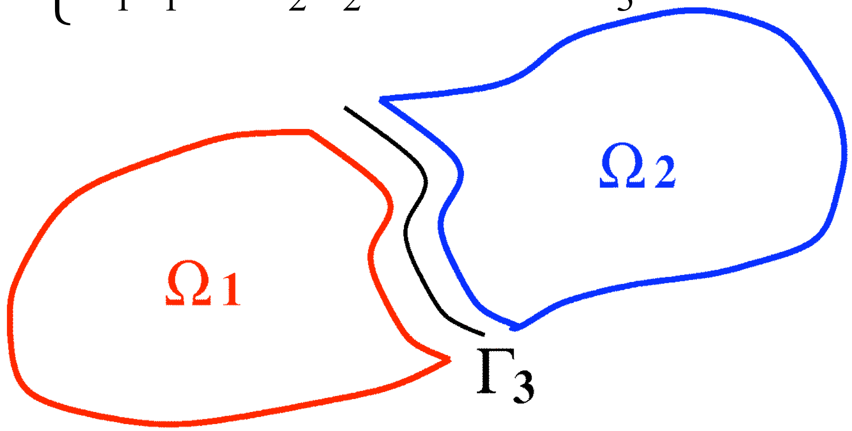
- Interface conditions

$$\begin{cases} u_1 = u_2 \text{ on } \Gamma_3 \\ \sigma_1 n_1 + \sigma_2 n_2 = 0 \ \text{ on } \Gamma_3 \end{cases}$$



- FETI method :

$$\begin{cases} \sigma_1 n_1 = -\sigma_2 n_2 = \lambda \\ g = u_1 - u_2 = 0 \end{cases}$$

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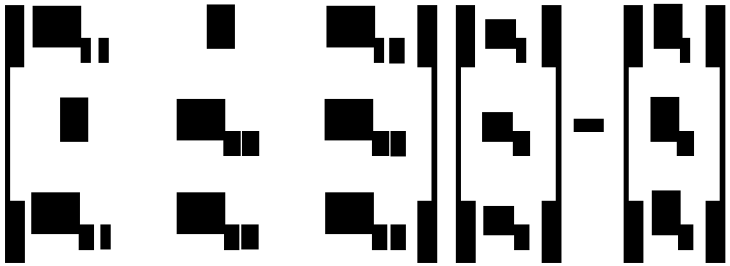
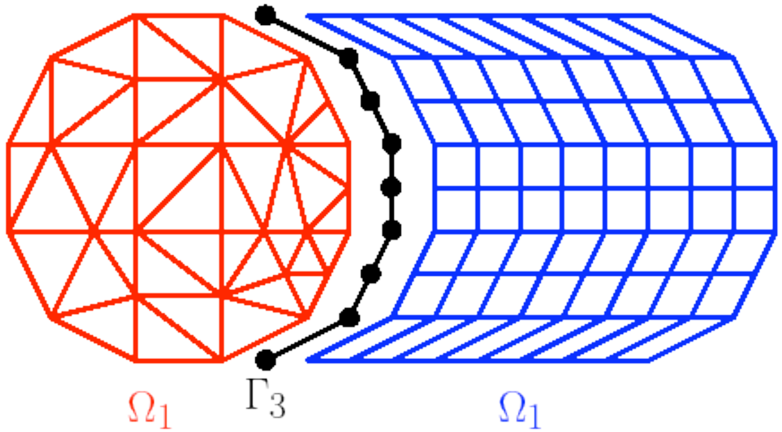
FETI : discrete model

- Global system of equations





- Local systems of equations: Neumann boundary conditions

- Interface problem

$$g = x_3^{(1)} - x_3^{(2)} = 0$$

FETI : algebraic formulation

- Local systems of equations

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- Elimination of inner variables

[illegible]

- FETI interface problem

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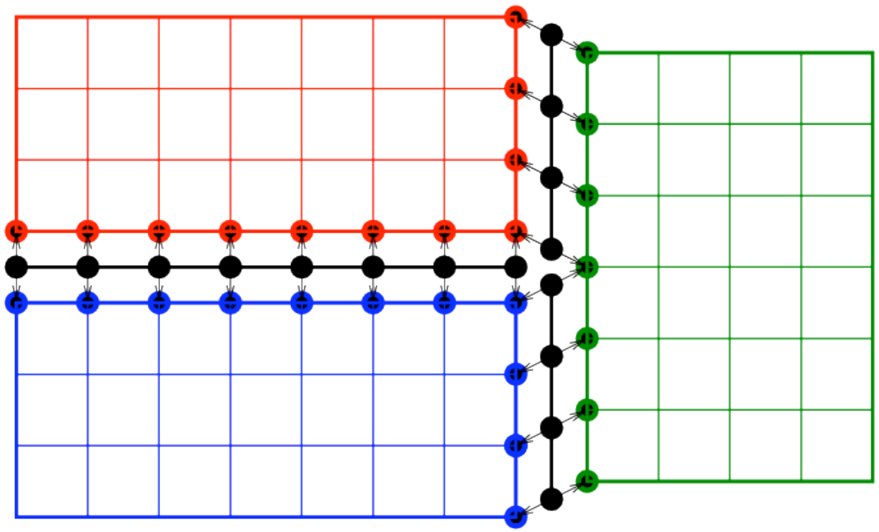
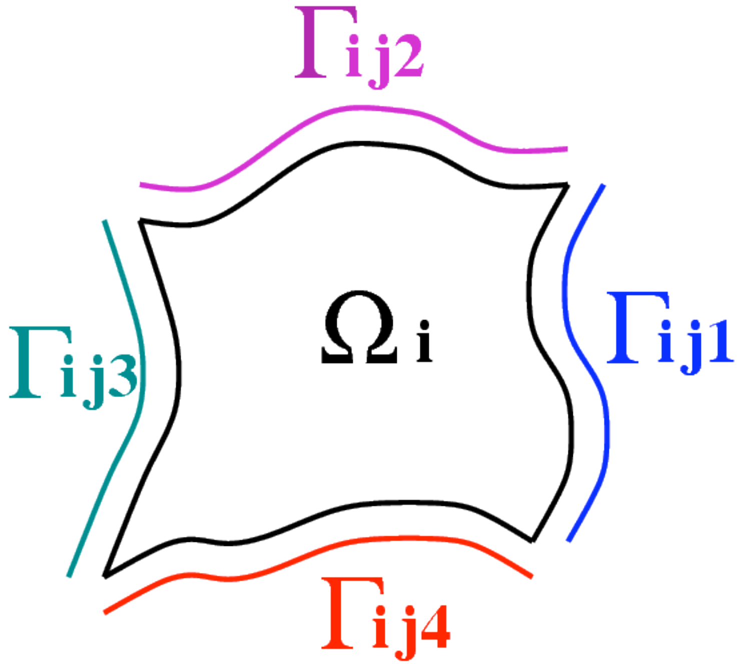
Easy interface management

- No multiple interface variables





- Uncoupled computation of jump of interface displacements

Optimal local preconditioner

- FETI condensed interface operator
- Optimal local preconditioner (convergence independent of mesh size)
- Computation of preconditioned gradient
- Solution of Dirichlet problem in each subdomain

Lumped local preconditioner

- FETI condensed interface operator

- Optimal local preconditioner

- Lumped local preconditioner

- Inexpensive, very efficient for moderate number of subdomains

Rigid body motions

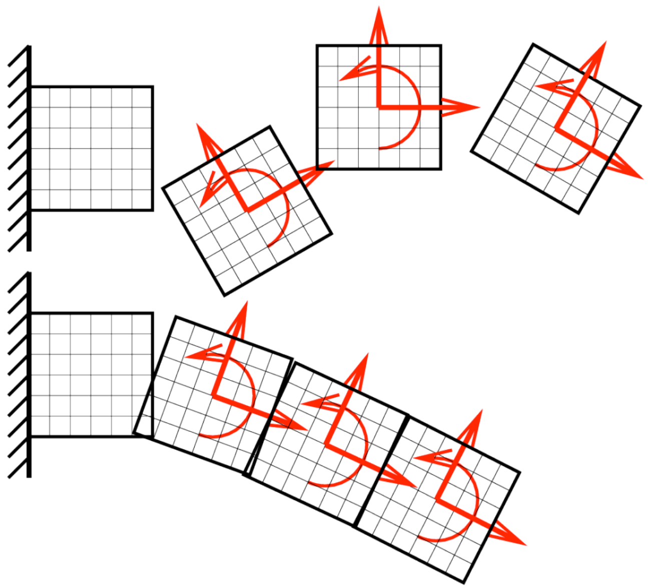
- Local Neumann problem ill posed
- Rigid body motions

- Mixed condensed interface problem solved via projection

- Coupling of local rigid body motions

Coarse grid preconditioner

- Projection for rigid body motion
  ⇒ computation of local rigid body motions that minimize interface jump
  ⇒ solution of global problem whose unknowns are rigid body motions



- Global coarse grid preconditioner
  ⇒ convergence independent upon number of subdomains

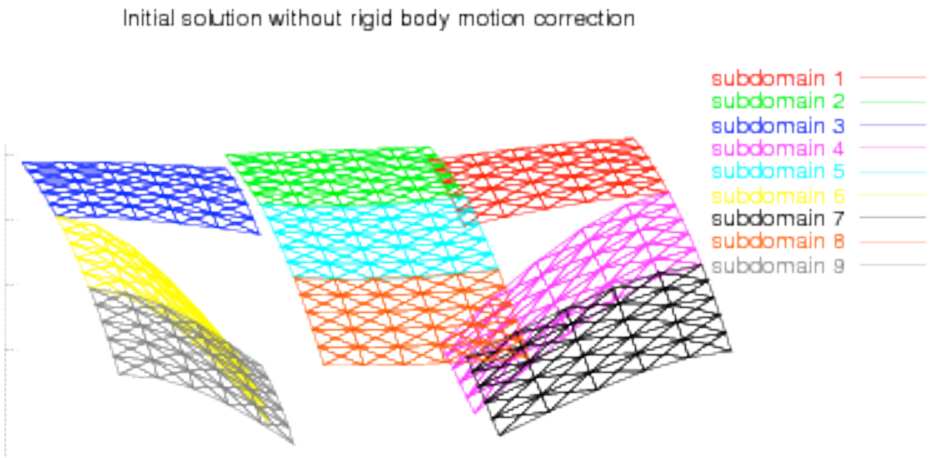
FETI-2

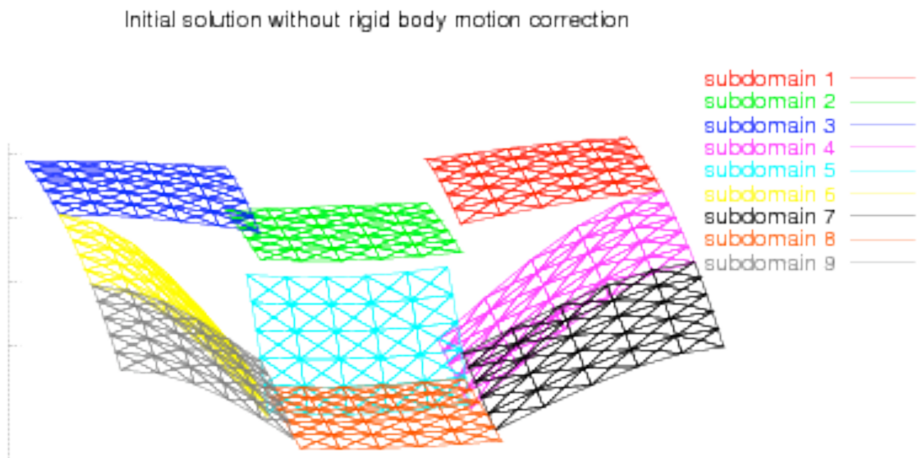
- Second level coarse preconditioner for FETI
- Forming and solving directly interface problem for selected nodes (corners for shell elements)
- Forming and solving directly interface problem for small set of trial interface modes (averaging modes = constant in one space direction, on edges or faces)

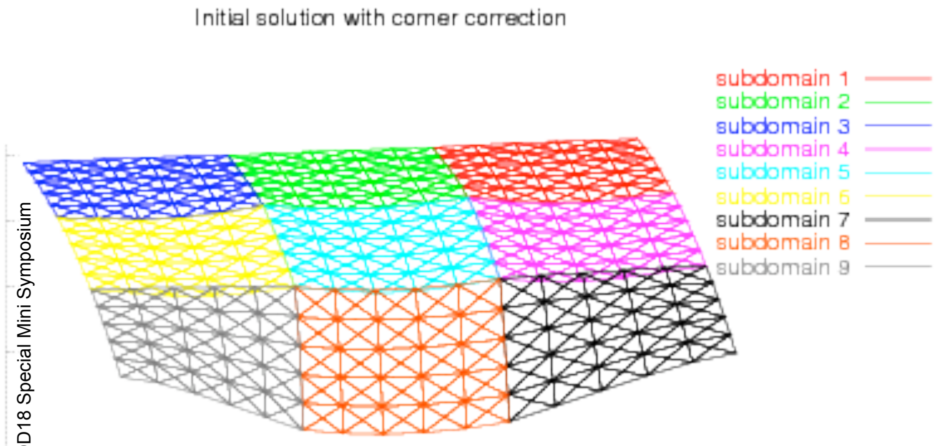
- Complex implementation due to rigid body projection
- Although forming the coarse problems requires only solution of local problem for local + neighbors modes, the coarse operator is a full matrix

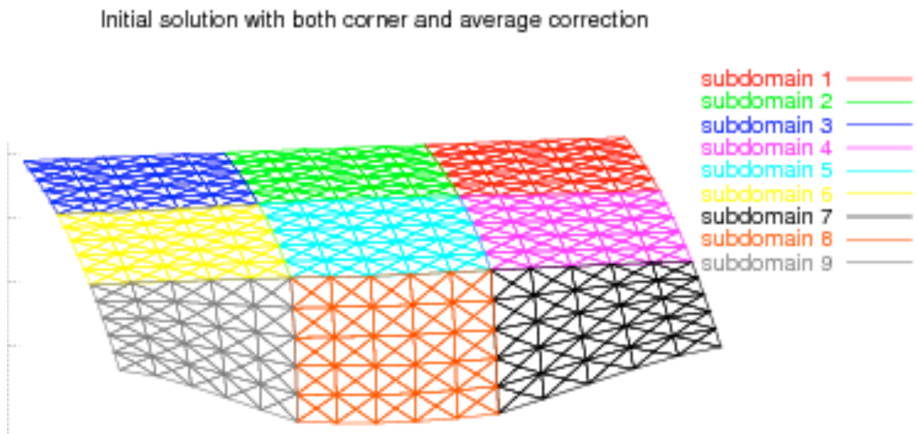
- A very efficient and simple methodology to build coarse grid preconditioners when there are no rigid body motions

FETI-2 for shell elements









FETI-DP

- How to get rid of rigid body motions to get sparse second-level coarse grid preconditioner?
- Add Dirichlet boundary conditions in subdomains

  ⇒ treat a small number of interface nodes in each subdomain in a primal way (Schur complement)
- In 2D topology (shell), corners
- Mixed variables on the interface

  ⇒ mostly dual, interaction forces

  ⇒ occasionally primal, displacements

- But the interface problem is mixed now

  ⇒ eliminate primal variables in second-level (sparse) coarse grid preconditioner

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YADP-FETI

- Mixed problems : Stokes equation, incompressible elasticity

- $\sigma.n$ and pn are forces
- Take $\sigma.n$ and p as interface variables

  ⇒ Neumann boundary conditions for displacements (dual)

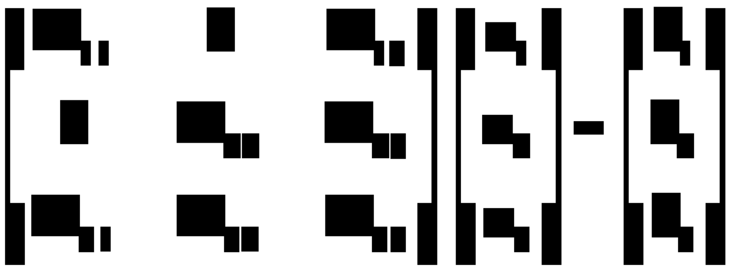
  ⇒ Dirichlet boundary conditions for pressure (primal)
- Local preconditioner : Dirichlet for displacements and Neumann for pressure
- Two projections

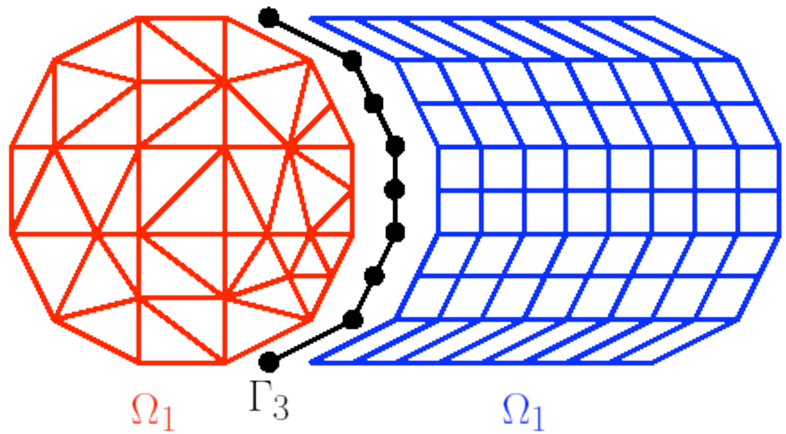
  ⇒ rigid body motions for the interface problem

  ⇒ constant pressure modes for the preconditioner (treated like in balanced method)

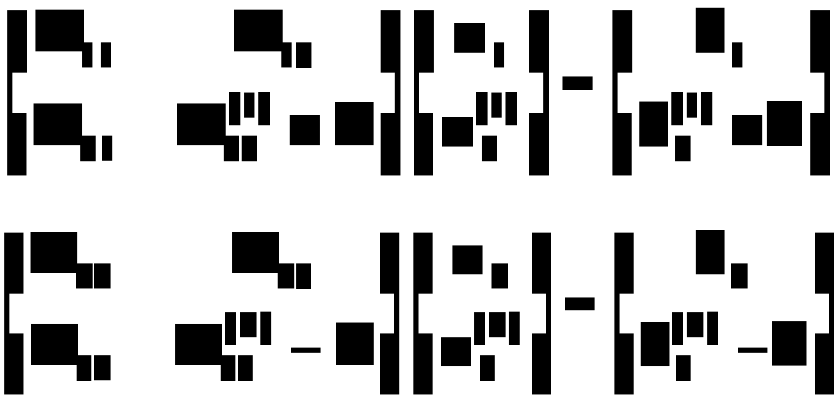
Arbitrary splitting of the global matrix on interface

- Global system of equations





- Local systems of equations : generalized Robin boundary conditions



- Interface problem, still

$$g = x_3^{(1)} - x_3^{(2)} = 0$$

FETI-H

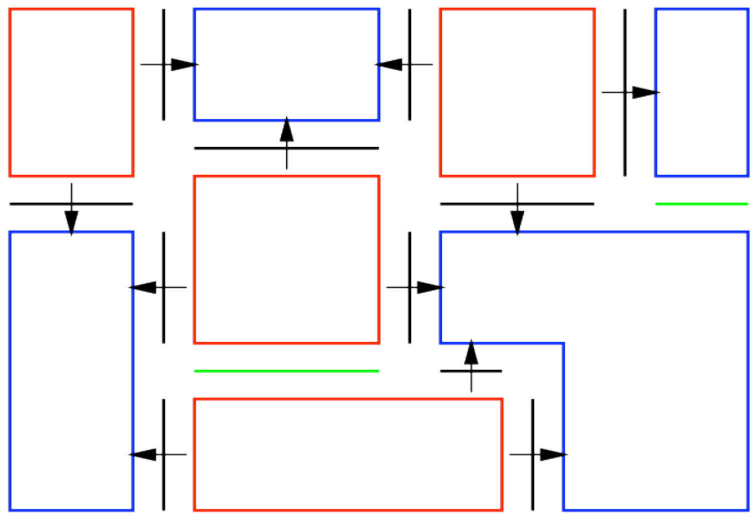
- For Helmholtz equation

$$-\Delta u - k^2 u = f$$

- Augmented matrix associated with inner or outer first order approximate transparent boundary condition

$$\frac{\partial u}{\partial n} \pm iku = \lambda$$

- Well posed local problem if all conditions are inward or outward
- Coloring of subdomains
- Mixing of absorbing and non absorbing interfaces



FETI-2LM method

- Global system of equations

[illegible]

- Local system of equations

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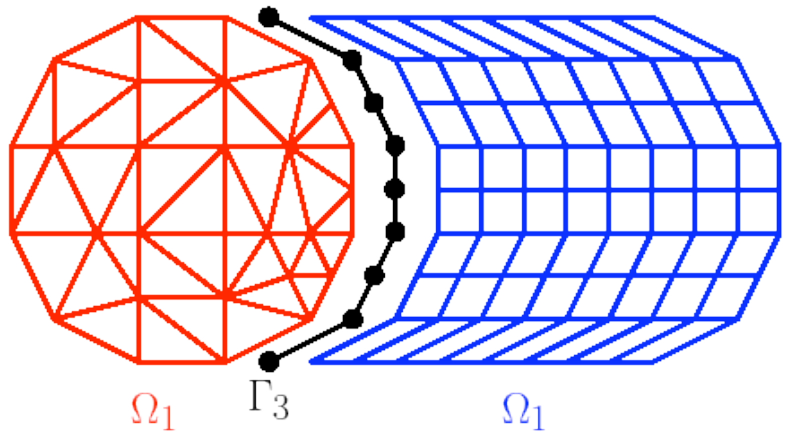

- Two interface conditions

$$\begin{cases} x_3^{(1)} = x_3^{(2)} \\ K_{31}x_1 + K_{32}x_2 + K_{33}^{(1)}x_3^{(1)} + K_{33}^{(2)}x_3^{(2)} = b_3 \\ \Leftrightarrow \lambda_1 + \lambda_2 - k_1 x_3^{(1)} - k_2 x_3^{(2)} = 0 \end{cases}$$

[illegible]

Condensed interface problem

- Local static condensation

[illegible]

[illegible]

- Matrix of interface problem

$$\begin{pmatrix} I & I-(k_1+k_2)(S^{(2)}+k_2)^{-1} \\ I-(k_2+k_1)(S^{(1)}+k_1)^{-1} & I \end{pmatrix}$$

Optimal interface connection conditions

- Optimal interface conditions

[illegible]

- Optimal conditions = static condensation on interface of remaining structure

- Interpretation via local static condensation in global system of equations

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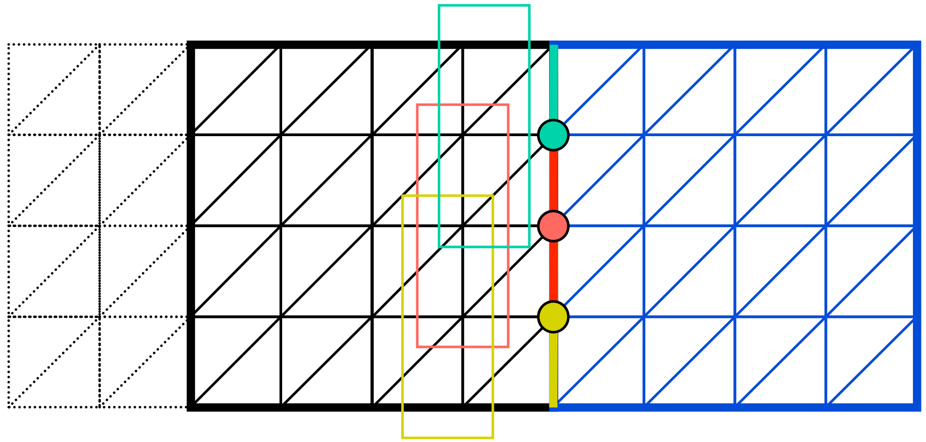
Main features of the method

- Well posed local problems, even with irregular mesh splitting
- Convergence in *p -1* iterations in case of one-way splitting into *p* subdomains
- Issue : computation of exact optimal operator impossible (Schur complement)
- Approximation required

Sparse approximation of Schur complement

- Local condensation on small patches
- Weighted assembling



- Purely algebraic approach
- Black box implementation
- Works for any kind of element
- Very fast convergence for highly heterogeneous problems

Conclusion

- Variety of methods depending on what problem to be solved
- Except for primal interface variables in DP methods, uncoupled interfaces
- General methodology of FETI-2 coarse grid preconditioner applies to all methods
- Choice of coarse space easy for coercive problems
- More difficult for wave propagation problems without absorbing terms
- FETI-2LM applies to wide range of problems
- All these methods can be used for non matching grids with mortar elements