Optimized Schwarz Methods

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Outline

1. Why working on DDM?
2. Schwarz method (1860)
3. Two families of methods

   Schur Complement type methods (Neumann-Neumann, FETI, BDDC, FETI-DP, ...

   Optimized Schwarz Methods
4. Conclusion and Open problems

Why working on DDM: a personal view

- a Matter of Taste

  Interplay between linear algebra and partial differential equations

- Background

  Optimized Schwarz methods are strongly related to parabolic approximations of PDEs and to Absorbing boundary conditions (my PhD subjects)

- Generality

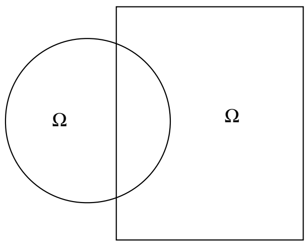
  A transverse tool (no specific application field) with regular meetings

- Future

  The feeling that DDMs are (part of) the future of scientific computing in connection with parallel computing.

The First Domain Decomposition Method

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Additive Schwarz Method : $(u_1^n, u_2^n) \rightarrow (u_1^{n+1}, u_2^{n+1})$ with

$$-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1 \qquad\qquad -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2$$
$$u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \qquad\qquad u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$
$$u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. \qquad\qquad u_2^{n+1} = u_1^{n(+1)} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.$$

Parallel algorithm, converges but very slowly

Matching interface conditions are of the Dirichlet type

Improvement will come from introducing Neumann or more general boundary conditions+ Krylov methods in replacement of the fixed point algorithm.

First family of modern methods : Schur Complement methods – Substructuring formulation

Consider a non overlapping decomposition of the domain Ω into Ω_1 and Ω_2 and Dirichlet BVP in each subdomain with $u|_\Gamma$ as a Dirichlet data

$$-\Delta(u_i) = f \quad \text{in } \Omega_i,$$
$$u_i = u|_\Gamma \quad \text{on } \Gamma, \qquad u_i = 0 \quad \text{sur } \partial\Omega_i \backslash \Gamma.$$

The jump of the normal derivative across the interface is a function of f and $u|_\Gamma$

$$\mathcal{S}(f, u|_\Gamma) = \frac{1}{2}\left(\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}\right)|_\Gamma$$

The substructured interface problem reads : Find $u|_\Gamma$ s.t.

$$\mathcal{S}(0, u|_\Gamma) = -\mathcal{S}(f, 0)$$

The corresponding discretized problem is solved by a Krylov type method such as CG, GMRES, BICGSTAB, QMR, ...

- Gain: if $\kappa(-\Delta_h) = O(1/h^2)$, then $\kappa(\mathcal{S}_h) = O(1/h)$. At the expense of a costly matrix-vector product.
- Extension : Find a good preconditioner $\mathcal{T}_h$ s.t. $\kappa(\mathcal{T}_h\,\mathcal{S}_h) \simeq O(1)$. The Neumann-Neumann preconditioner is based on solving Neumann problems in the subdomains.

FETI method : Substructured problem in terms of the normal derivative of the solution on the interface, preconditioned by Dirichlet problems.

Other possible improvement: Other Interface Conditions

(P.L. Lions, 1988)

$$
\begin{aligned}
&-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1,\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&(\frac{\partial}{\partial n_1} + \alpha)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \alpha)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},
\end{aligned}
$$

(n_1 and n_2 are the outward normal on the interface)

$$
\begin{aligned}
&-\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2,\\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
&(\frac{\partial}{\partial n_2} + \alpha)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \alpha)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

with $\alpha > 0$. Overlap is not necessary for convergence.

Extended to the Helmholtz equation (B. Desprès, 1991): the first iterative solver to Helmholtz.

a.k.a Two-Lagrange Multiplier FETI Method, FETI-2LM 1998.

- Gain: Much faster convergence, no need for overlaps
- Extensions:

  Find even better interface conditions (Optimized Scwharz methods)

  introduce Krylov type methods in place of the above fixed point algorithm

Optimized Schwarz Methods

1. Optimal Schwarz Methods
2. Optimized Schwarz Methods
3. Application to the Helmholtz equation
4. Algebraic Optimized Schwarz Methods
5. Conclusion

Optimal Schwarz Methods

(Hagstrom, 1988)
Constant coefficient Advection-Diffusion equation on a domain decomposed into two subdomains.

$$
\begin{aligned}
&(\vec{a}\nabla - \nu\Delta)(u_1^{n+1}) = f \quad \text{in } \Omega_1,\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&(\frac{\partial}{\partial n_1} + \mathcal{B}_1)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \mathcal{B}_1)(u_2^{n}) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},
\end{aligned}
$$

$$
\begin{aligned}
&(\vec{a}\nabla - \nu\Delta)(u_2^{n+1}) = f \quad \text{in } \Omega_2,\\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
&(\frac{\partial}{\partial n_2} + \mathcal{B}_2)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \mathcal{B}_2)(u_1^{n}) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

where $\mathcal{B}_i$, $i = 1, 2$ are defined via a Fourier transform along the interface
Convergence in two iterations

Optimal Schwarz Methods

Let us consider the problem

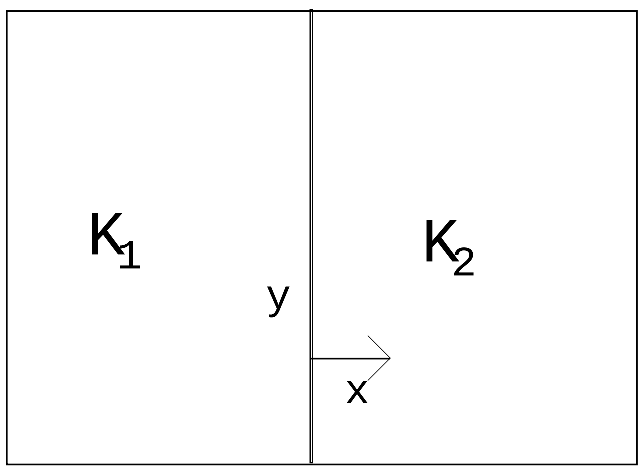
$$\mathcal{L}_i(P_i) = f \quad \text{in } \Omega_i, \quad i = 1, 2$$

$$P_1 = P_2 \quad \text{on } \Gamma_{12},$$

$$\kappa_1 \frac{\partial P_1}{\partial n_1} + \kappa_2 \frac{\partial P_2}{\partial n_2} = 0 \quad \text{on } \Gamma_{12}.$$

where

$$\mathcal{L}_i = \eta_i - div(\kappa_i \nabla \dot{)}$$



Optimal Schwarz Methods

Let

$$u_i = \kappa_i \nabla P_i$$

Let us consider a Schwarz type method:

$$\mathcal{L}_1(P_1^{n+1}) = f \quad \text{in } \Omega_1 \qquad\qquad \mathcal{L}_2(P_2^{n+1}) = f \quad \text{in } \Omega_2$$

$$P_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \qquad\qquad P_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$

$$u_1^{n+1}.\vec{n}_1 + \mathcal{B}_1(P_1^{n+1}) \qquad\qquad u_2^{n+1}.\vec{n}_2 + \mathcal{B}_2(P_2^{n+1})$$

$$= -u_2^n.\vec{n}_2 + \mathcal{B}_1(P_2^n) \quad \text{on } \Gamma_1 \qquad\qquad = -u_1^n.\vec{n}_1 + \mathcal{B}_2(P_1^n) \quad \text{on } \Gamma_2$$

We take

$$\mathcal{B}_1 = DtN_2.$$

and have convergence in two iterations.

The optimal interface conditions are Exact Absorbing Boundary Conditions

Optimal Schwarz Methods

We introduce the DtN (Dirichlet to Neumann) map (a.k.a. Steklov-Poincaré):

$$\text{Let } P_0 : \Gamma_{12} \longrightarrow \mathbb{R} \tag{1}$$

$$DtN_2(P_0) \equiv \kappa_2 \frac{\partial}{\partial n_2}(P)_{|\Gamma_{12}}$$

where n_2 is the outward normal to $\Omega_2 \backslash \bar{\Omega}_1$ and P satisfies the following boundary value problem:

$$\mathcal{L}(P) = 0 \text{ in } \Omega_2$$

$$P = 0 \text{ on } \partial\Omega_2 \backslash \Gamma_i$$

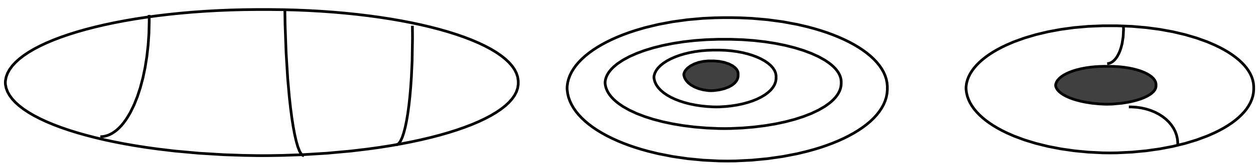
$$P = P_0 \text{ on } \Gamma_{12}.$$

Recall

$$\mathcal{B}_1 = DtN_2.$$

OptimalSchwarz Methods

(Rogier, de Sturler and N., 1993)
The result can be generalized to variable coefficients operators and a decomposition of the domain Ω in more than two subdomains.
For the following geometries,



one can define interface conditions such as to have convergence in a number of iterations equals to the number of subdomains.
For arbitrary decompositions, negative conjectures have been formulated (F. Nier, *Séminaire X-EDP*, 1998).

Optimized Interface Conditions

The Dirichlet to Neumann map (DtN) is not a partial differential operator :

1. it is non local
2. no explicit formula in the general case

It is approximated by a partial differential operator

$$DtN \simeq \alpha_{opt} - \frac{\partial}{\partial \tau}(\gamma_{opt} \frac{\partial}{\partial \tau})$$

minimizing the convergence rate using Fourier transform as an essential tool. The resulting interface conditions are called optimized of order 2 (opt2) interface conditions.
Work of C. Japhet on the convection-diffusion equation (PhD thesis, 1998)

More relevant than classical approximate absorbing boundary conditions.

Application: the Helmholtz Equation

Joint work with M. Gander and F. Magoulès

SIAM J. Sci. Comp., 2002.

We want to solve

$$-\omega^2 u - \Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

The relaxation algorithm is : $(u_1^p, u_2^p) \to (u_1^{p+1}, u_2^{p+1})$ with $(i \neq j, i = 1,2)$

$$(-\omega^2 - \Delta)(u_i^{p+1}) = f \quad \text{in } \Omega_i$$
$$(\frac{\partial}{\partial n_i} + \mathcal{S})(u_i^{p+1}) = (-\frac{\partial}{\partial n_j} + \mathcal{S})(u_j^p) \quad \text{on } \Gamma_{ij}.$$
$$u_i^{p+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega$$

The operator $\mathcal{S}$ has the form

$$\mathcal{S} = \alpha - \gamma \frac{\partial^2}{\partial \tau^2} \qquad \alpha, \gamma \in \mathbb{C}$$

Application: the Helmholtz Equation

By choosing carefully the coefficients α and γ, it is possible to optimize the convergence rate of the iterative method which in the Fourier space is given by

$$\rho(k;\alpha,\gamma) \equiv \begin{cases} \left| \dfrac{I\sqrt{\omega^2-k^2}-(\alpha+\gamma k^2)}{I\sqrt{\omega^2-k^2}+(\alpha+\gamma k^2)} \right| & \text{if } \omega \;> |k| \; (I^2=-1) \\[2em] \left| \dfrac{\sqrt{k^2-\omega^2}-(\alpha+\gamma k^2)}{\sqrt{k^2-\omega^2}+(\alpha+\gamma k^2)} \right| & \text{if } \omega < |k| < 1/h \end{cases}$$

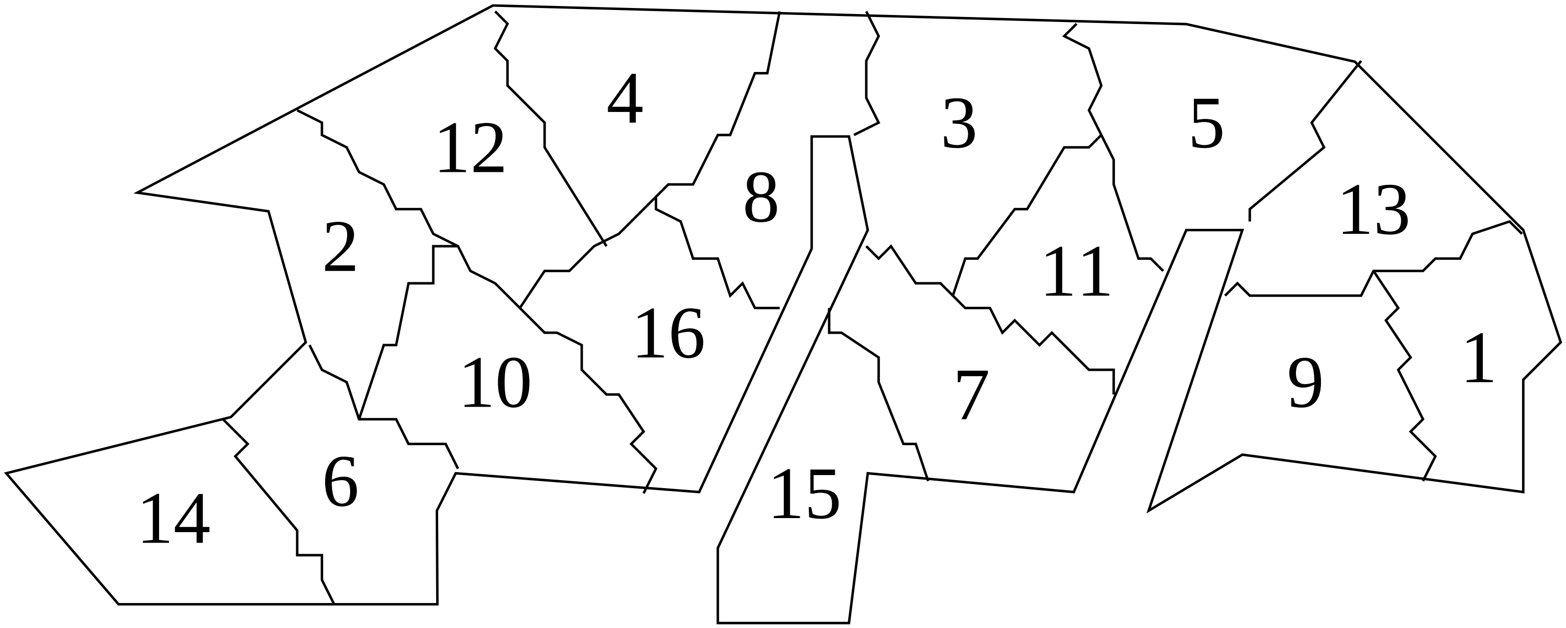
Finally, we get analytic formulas for α and γ (h is the mesh size):

$$\alpha_{opt} = \alpha(\omega, h) \text{ and } \gamma_{opt} = \gamma(\omega, h),$$

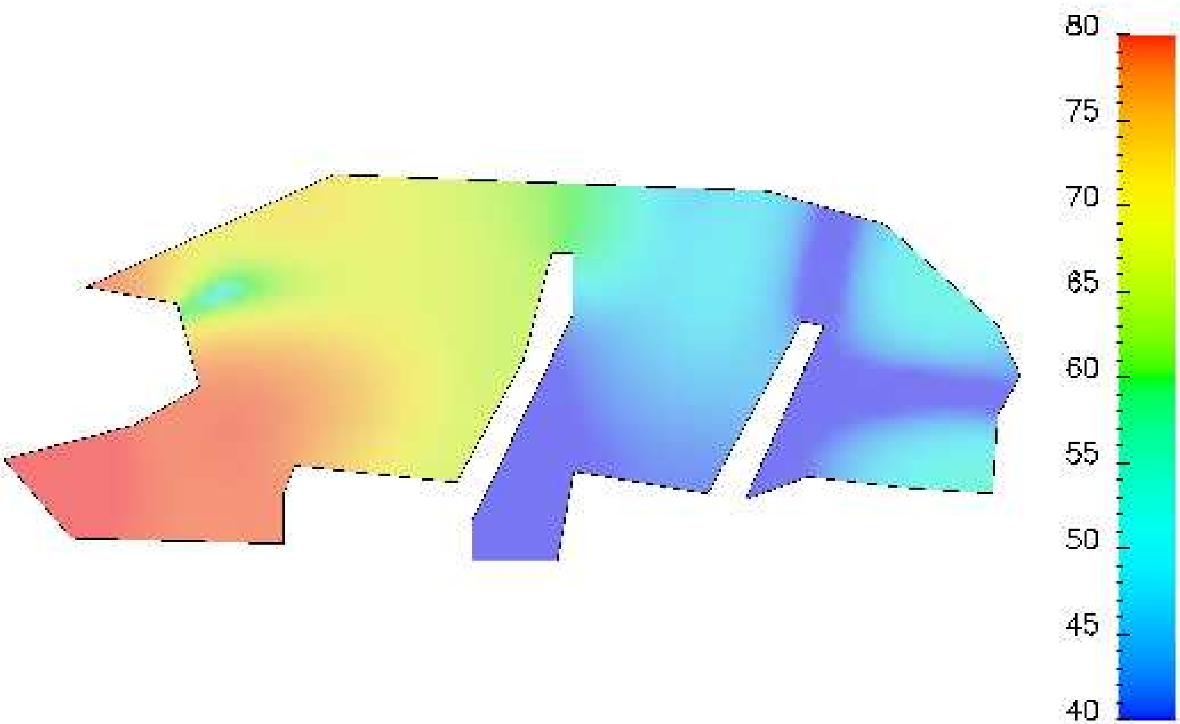
Moreover, a Krylov method (GC, GMRES, BICGSTAB, ...) replaces the fixed point algorithm.

Optimized Schwarz method for the Helmholtz Equation

Numerical Results: Acoustic in a Car



Numerical Results: Acoustic in a Car



Optimized Schwarz method for the Helmholtz Equation

Numerical Results

Acoustic in a Car : Iteration Counts for various interface conditions

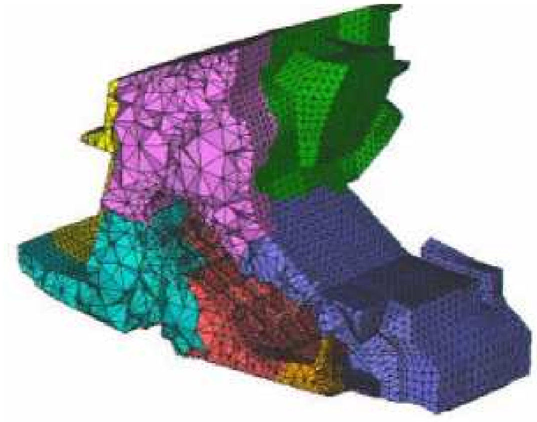
N_s	ABC 0	ABC 2	Optimized
2	16 it	16 it	9 it
4	50 it	52 it	15 it
8	83 it	93 it	25 it
16	105 it	133 it	34 it

ABC 0: Absorbing Boundary Conditions of Order 0 $(\partial_n + I\omega)$

ABC 2: Absorbing Boundary Conditions of Order 2 $(\partial_n + I\omega - 1/(2I\omega)\partial_{yy})$

Optimized: Optimized Schwarz Methods

Motor compartment



Frequency	Number of sub-domains	Order Zero		Order Two	
		Taylor	Optimized	Taylor	Optimized
600	2	451	205	453	147
600	4	573	287	625	186
600	8	715	355	803	237
800	2	447	221	445	146
800	4	647	323	733	212
800	8	1069	531	1105	354

Table 5: Number of iterations for different transmission conditions, frequencies values and numbers of sub-domains for the engine compartment problem.

Optimal Schwarz Methods at the matrix level

When a finite element method, for instance, is used it yields a linear system of the form $AU = F$, where F is a given right-hand side and U is the set of unknowns.
Corresponding to a domain decomposition, the set of unknowns U is decomposed into interior nodes of the subdomains U_1 and U_2, and to unknowns, U_Γ, associated to the interface Γ.
This leads to a block decomposition of the linear system

$$\begin{pmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_\Gamma \\ U_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_\Gamma \\ F_2 \end{pmatrix} . \tag{2}$$

Optimal Schwarz Methods at the matrix level

The DDM method reads:

$$\begin{pmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + S_2 \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_{\Gamma,1}^{n+1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_\Gamma + S_2 U_{\Gamma,2}^n - A_{\Gamma 2} U_2^n \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + S_1 \end{pmatrix} \begin{pmatrix} U_2^{n+1} \\ U_{\Gamma,2}^{n+1} \end{pmatrix} = \begin{pmatrix} F_2 \\ F_\Gamma + S_1 U_{\Gamma,1}^n - A_{\Gamma 1} U_1^n \end{pmatrix} \tag{4}$$

where

$$S_1 = -A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$$

and

$$S_2 = -A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

Convergence in two iterations

Approximate Interface Condition at the matrix level

The matrices $S_1 = -A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$ and $S_2 = -A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$ are full interface matrices $(\Gamma \times \Gamma)$.

Cons

- Costly to compute
- The subdomain matrix is partly full

Approximate S_1 and S_2 by sparse matrices

1. e.g. via sparse approximations to A_{ii}^{-1}: SPAI
2. via local Schur complements on successive reduced "outer" domains, "patches", $(\gamma \times \delta)$ (Roux, 2003)

The first approach gives mild results. The second one is not better than using an overlap of depth δ but is cheaper.

Conclusion

- Both approaches (Neumann-Neumann and optimized Schwarz methods) are robust (thanks to Krylov methods).
- Neumann-Neumann, FETI, .. optimal but lacks generality
- optimized Schwarz methods are general but are more difficult to tune

Open problems

- Theory

  Convergence proof or condition number estimate in a general overlapping case

  proof of the Non existence of Optimal Interface Conditions for a general domain decomposition

- Algorithm

  Algebraic Optimized Interface Conditions

  Interplay between the Optimized Interface Conditions and a Coarse Grid (see Japhet, Nataf, Roux, 1998)

  Systems of PDEs (versus scalar PDEs)

Thanks !

How to use Krylov type methods ?

Schwarz method

with arbitrary interface conditions

$\mathcal{C}_1$ and $\mathcal{C}_2$

$$
\begin{aligned}
&\mathcal{L}(u_1^{n+1}) = f \quad \text{in } \Omega_1,\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&\mathcal{C}_1(u_1^{n+1}) = \mathcal{C}_1(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},\\
&\mathcal{L}(u_2^{n+1}) = f \quad \text{in } \Omega_2,\\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\
&\mathcal{C}_2(u_2^{n+1}) = \mathcal{C}_2(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1},
\end{aligned}
\iff
\begin{array}{c}
Sub(\mathcal{C})(\lambda) = b \text{ solved by Jacobi}\\
\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \lambda_i = \mathcal{C}_i(u_j),\ i \neq j\\
\Downarrow\\
\text{CG, GMRES, BICG, } \ldots
\end{array}
$$