Optimized Schwarz waveform relaxation algorithms

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Outline

Parallel processing of evolution problems

$$P(\partial_t, \partial_1, \cdots, \partial_d)u = f$$

- *Explicit schemes* : exchange of informations between processors at every time-step.

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  D. Chazan and W. Miranker ,69
  D. Amitai, A.Averbuch, S.Itzikowitz, M.Israeli, E. Turkel 93
  "A major obstacle to achieving significant speed-up on parallel machines is the overhead associated with synchronizing the concurrent processes. There are various reasons why certain processors will be ahead of the others, even when they are physically configured at the same speed. Among those I Random noise, 2 Load balancing Second, there is a delay period associated with the synchronization mechanism itself whether it is setting the semaphores in a shared memory environment or waiting on a message to arrive in a message passing environment".

Parallel processing of evolution problems

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- *Explicit schemes* : exchange of informations between processors at every time-step. $->$ asyncronous algorithms
  Cons: loose convergence, difficult to implement

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  Kuznetsov, 88, Meurant, 91, Cai, 91, Dryja, 91.
  Improves the condition number.
  See Quarteroni-Valli book.

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  Cons: uniform time-step.

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  Cons: need regular problems

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- *Schwarz waveform relaxation* Gander and Giladi-Keller 1997

Parallel processing of evolution problems

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  G. Horton et S. Vandewalle, 1993
- *Schwarz waveform relaxation* Gander and Giladi-Keller 1997
- *Multigrid in time* – > parareal algorithms J.L. Lions, Turinici, Maday.

Why Schwarz Waveform relaxation ?

flexibility

◇ can choose the space and time meshes independently in the subdomains – > local space-time refinement with time windows.

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- ♢ can use different numerical schemes in the subdomains,
- ♢ can even couple different models,

Why Schwarz Waveform relaxation ?

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- ◇ can use different numerical schemes in the subdomains,
- ◇ can even couple different models,
- ◇ adjust to underlying computing hardware.

Outline

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The ancestor

Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives;

PAR M. ÉMILE PICARD.

INTRODUCTION.

Considérons une équation du second ordre aux dérivées partielles de la forme

$$(1)\qquad A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}=F\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},x,y\right),$$

A, B, C dépendant seulement des deux variables indépendantes x et y. On peut, pour intégrer cette équation, avec des conditions aux limites déterminées, procéder de la manière suivante par approximations successives. Nous mettons dans le second membre une fonction quelconque u_1 de x et y, et formons l'équation

$$\Delta u_2=F\left(u_1,\frac{\partial u_1}{\partial x},\frac{\partial u_1}{\partial y},x,y\right)$$

$\left(\text{en posant ici, pour abréger, } \Delta u = A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}\right)$. Concevons qu'on intègre cette équation en u_2, en se donnant certaines conditions aux limites, qui, nous le supposons, déterminent complètement une intégrale que nous désignerons par u_2. On formera ensuite

Prenons tout d'abord une seule équation du premier ordre

$$\frac{dy}{dx}=f(y,x);$$

on peut établir ainsi le théorème fondamental relatif à l'existence de l'intégrale de cette équation, prenant pour $x=x_0$ la valeur $y=y_0$. On considérera, à cet effet, les équations

$$\frac{dy_1}{dx}=f(y_0,x),$$
$$\frac{dy_2}{dx}=f(y_1,x),$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$\frac{dy_n}{dx}=f(y_{n-1},x),$$

en effectuant chacune des quadratures, de façon que pour $x=x_0$ on ait $y_n=y_0$. Il s'agit d'établir que y_n tend, pour n infini, vers une limite y qui représentera l'intégrale cherchée, pourvu d'ailleurs que x reste dans le voisinage de x_0. Nous faisons sur la fonction $f(y,x)$, l'hypothèse qu'elle est continue et définie pour les valeurs de x et de y comprises respectivement entre x_0-a et x_0+a d'une part, puis y_0-b et y_0+b d'autre part; de plus, on peut déterminer une constante positive k, telle que

$$|f(y_2,x)-f(y_1,x)|<k|y_2-y_1|$$

et nous supposons la fonction et les variables réelles.

Soit M le module maximum de $f(y,x)$ quand x et y restent entre les limites indiquées. On aura

$$y_1=\int_{x_0}^{x}f(y_0,x)\,dx+y_0.$$

Soit ρ une quantité au plus égale à a; y_1 restera dans les limites voulues si

$$M\rho<b,$$

Waveform relaxation. Lelarasmee 1982, Nevanlinna, Vandevalle

Review: Burrage *et al*, Appl. Num. Math. 1996.

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, \cdots, y_p),$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, \cdots, y_p)$$
$$\frac{dy_j}{dt} = f_j(t, y_1, y_2, \cdots, y_p)$$
$$\frac{dy_p}{dt} = f_p(t, y_1, y_2, \cdots, y_p)$$

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\frac{dy_p}{dt} &= f_p(t, y_1, y_2, \cdots, y_p)
\end{aligned}
$$

Approximations successives

$$
\begin{aligned}
\frac{dy_1^{(k+1)}}{dt} &= f_1(t, y_1^{(k)}, y_2^{(k)}, \cdots, y_p^{(k)}),\\
\frac{dy_2^{(k+1)}}{dt} &= f_2(t, y_1^{(k)}, y_2^{(k)}, \cdots, y_p^{(k)})\\
\frac{dy_j^{(k+1)}}{dt} &= f_j(t, y_1^{(k)}, y_2^{(k)}, \cdots, y_p^{(k)})\\
\frac{dy_p^{(k+1)}}{dt} &= f_p(t, y_1^{(k)}, y_2^{(k)}, \cdots, y_p^{(k)})
\end{aligned}
$$

$$\|y^{(k+1)} - y\|_\infty \leq \frac{L^k (T - t_0)^k}{k!} \|y^{(0)} - y\|_\infty$$

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\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \cdots, y_p),\\
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\end{aligned}$$

Jacobi

$$\begin{aligned}
\frac{dy_1^{(k+1)}}{dt} &= f_1(t, y_1^{(k+1)}, y_2^{(k)}, y_j^{(k)}, \cdots, y_p^{(k)}),\\
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\end{aligned}$$

Gauss-Seidel

$$\begin{aligned}
\frac{dy_1^{(k+1)}}{dt} &= f_1(t, y_1^{(k+1)}, y_2^{(k)}, \cdots, y_j^{(k)}, y_p^{(k)}), \\
\frac{dy_2^{(k+1)}}{dt} &= f_2(t, y_1^{(k+1)}, y_2^{(k+1)}, \cdots, y_j^{(k)}, y_p^{(k)}) \\
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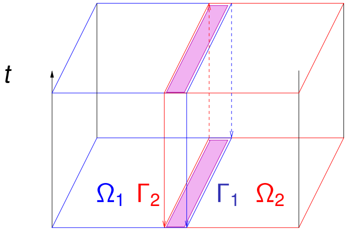
\end{aligned}$$

The Schwarz waveform relaxation algorithm



$$\left\{\begin{array}{rcll} \mathcal{L}u_1^{k+1} & = & f & \text{in } \Omega_1 \times (0,T) \\ u_1^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_1 \\ u_1^{k+1} & = & u_2^k & \text{on } \Gamma_1 \times (0,T) \end{array}\right.$$

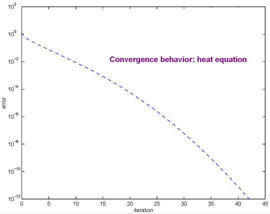
$$\left\{\begin{array}{rcll} \mathcal{L}u_2^{k+1} & = & f & \text{in } \Omega_2 \times (0,T) \\ u_2^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_2 \\ u_2^{k+1} & = & u_1^k & \text{on } \Gamma_2 \times (0,T) \end{array}\right.$$

Gander and Giladi-Keller 1997.
Heat equation and convection-diffusion equation.
Short and long time behavior.

Behavior of the Schwarz waveform relaxation algorithm

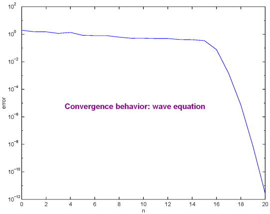
Heat equation

$$\partial_t u - \Delta u = 0$$



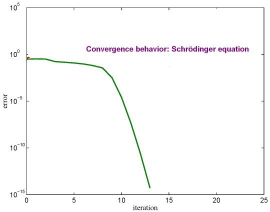
Wave equation

$$\partial_{tt} u - \Delta u = 0$$



Schrödinger equation

$$i\,\partial_t u + \Delta u = 0$$

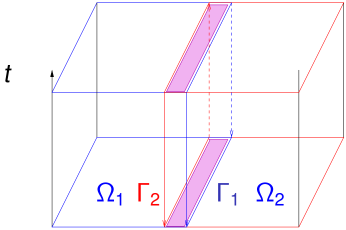


Outline

The Schwarz waveform relaxation algorithm

$$\mathcal{L}u := \partial_t u + (\boldsymbol{a} \cdot \nabla)u - \nu \Delta u + cu = 0 \text{ in } \Omega \times (0, T)$$

$$\nu > 0.$$



$$\begin{cases} \mathcal{L}u_1^{k+1} = f & \text{in } \Omega_1 \times (0,T) \\ u_1^{k+1}(\cdot,0) = u_0 & \text{in } \Omega_1 \\ u_1^{k+1} = u_2^k & \text{on } \Gamma_1 \times (0,T) \end{cases}$$

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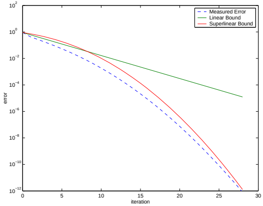
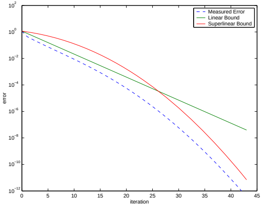
Properties

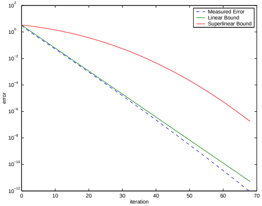
Superlinear convergence on short time interval + linear convergence on infinite time.



T=1



T=2.5



T=10

Mathematical tools: maximum principle and Fourier transform in time/transverse space variables.

The convergence rate depends only on the number of subdomains in higher order terms

coarse grid preconditioners are not necessary .

The Modified Schwarz algorithm

Jacobi or Gauss-Seidel way:

$$\mathcal{L}u := \partial_t u + (\boldsymbol{a}\cdot\boldsymbol{\nabla})u - \nu\Delta u + cu \text{ in } \Omega\times(0,T)$$

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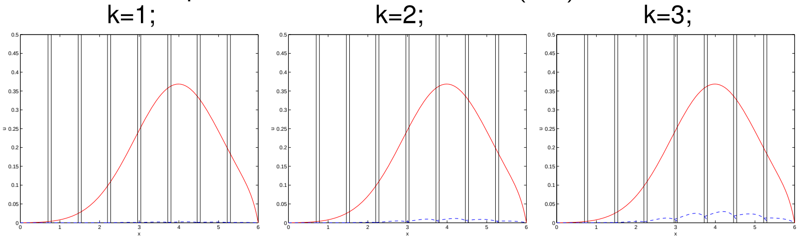
First attempt: Robin transmission condition

1D Numerical experiment $a = 1, \nu = 0.2, \Omega = (0, 6), T = 2.5, L = 0.08.$



With 2 subdomains: Gander, L.H, Nataf, DD 11, 1998.

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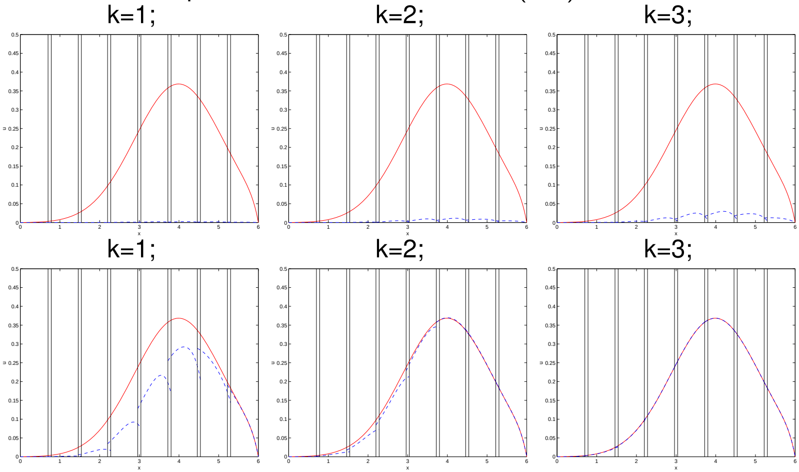
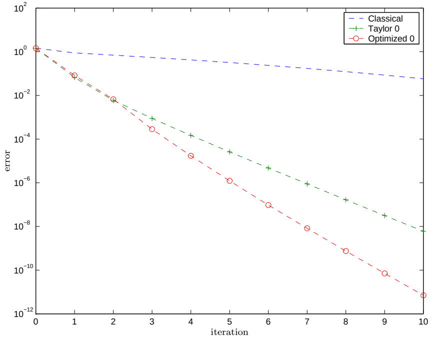


With 2 subdomains: Gander, L.H, Nataf, DD 11, 1998.

Comparison

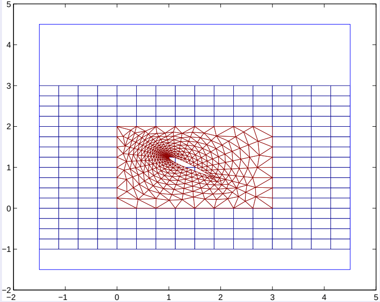
$$\mathcal{B}_j := \frac{\partial}{\partial n_j} - \boldsymbol{a} \cdot \boldsymbol{n}_j + pI$$

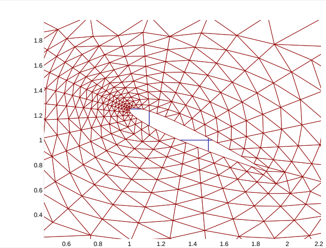


Two dimensions : coupling different numerical methods

The heat bubble hitting an airfoil

Advection-diffusion equation, Coupling through `Corba`
P.d'Anfray, J. Ryan, L.H. M2AN 2002





Two dimensions : coupling different numerical methods

Programming

- NO OVERLAP
- F.E in Ω_1, $F.D$ in Ω_2,
- Write the interface problem,
- solve by Krylov,

Two dimensions : coupling different numerical methods

Results for one time window

Steady algorithm

```
do time iterations 1:N
do Krylov iterations
residual vectors =
size of interface
```

Unsteady algorithm

```
do Krylov iterations
do time iterations 1:N
residual vectors =
size of interface x N
```

Two dimensions : coupling different numerical methods

Results for one time window

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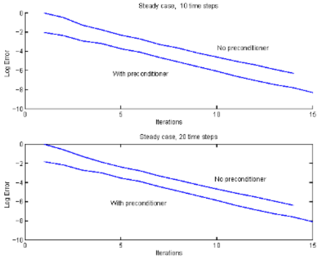


Figure 12: Effect of the preconditionner

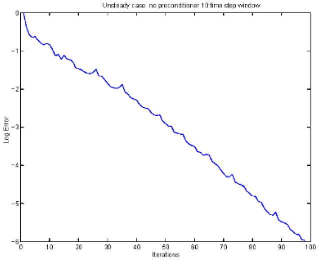


Figure 13: Unsteady case

Generalisation

optimal Schwarz Waveform relaxation WITH OR WITHOUT overlap.

Boundary operators

$$\mathcal{B}_1 u := \partial_x u - \frac{\boldsymbol{a}\cdot\boldsymbol{n} - \mathbf{p}}{2\nu} u + \mathbf{q}(\partial_t + \cdot\nabla u - \nu\Delta_S u)$$

THEOREM

For $p, q > 0$, $p > \frac{a^2}{4\nu} q$, the algorithm is well-posed in suited Sobolev spaces and converges with and without overlap.

Well-posedness and convergence

The case of half-spaces and constant coefficients

Fourier transform in time and transverse space

$$\delta(z) = a^2 + 4\nu c + 4\nu z, z = i(\omega + \boldsymbol{b} \cdot \boldsymbol{k}) + \nu|\boldsymbol{k}|^2,$$

Convergence factor

$$\rho(\omega, k, P, L) = \left(\frac{P - \delta^{1/2}}{P + \delta^{1/2}}\right)^2 e^{-2\delta^{1/2}L/\nu}$$

$$\widehat{e_j^{k+2}}(\omega, 0, k) = \rho(\omega, k, P, L)\widehat{e_j^{k}}(\omega, 0, k)$$

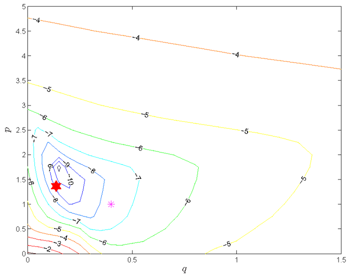
The nonoverlapping case

Energy estimates

Gander-Halpern 07, Bennequin-Gander-Halpern 08.

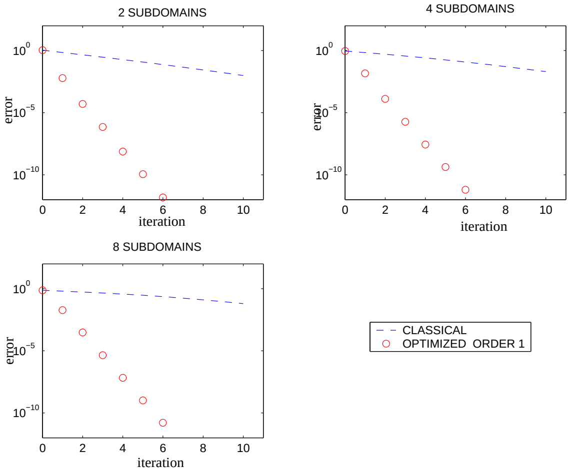
One dimension: influence of the parameters

steady credit. OO2: Optimized of order two, Caroline Japhet, PhD 1998.

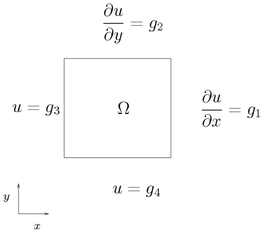


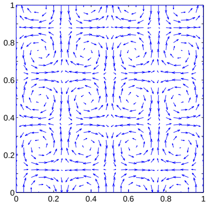
Error obtained running the algorithm with first order transmission conditions for 5 steps and various choices of p and q.
p^*, q^*: theoretical values ,
p^*, q^*: Taylor approximations.

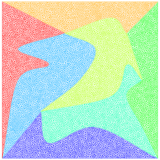
One dimension: comparison

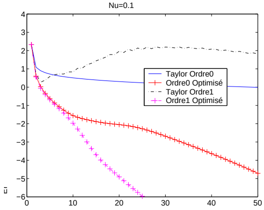


Robustness: rotating velocities

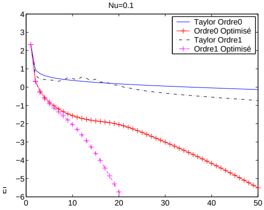




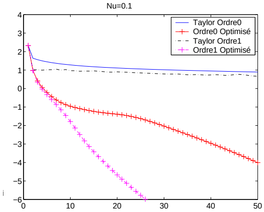




interface 0.3



interface 0.4



interface 0.5

Véronique Martin, PhD 2004. Loic Gouarin for the movie.

Optimization of the convergence factor

$$\delta(z) = a^2 + 4\nu c + 4\nu z, z = i(\omega + \boldsymbol{b} \cdot \boldsymbol{k}) + \nu|k|^2$$

$$\rho(z, P, L) = \left(\frac{P(z) - \delta^{1/2}(z)}{P(z) + \delta^{1/2}(z)}\right)^2 e^{-2\delta^{1/2}L}$$

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THEOREM

For any n, for $L = 0$ or sufficiently small, the problem has a unique solution characterized by an equioscillation property.

Asymptotic results

Example: overlapping case, $L \approx C\Delta x$, $\Delta t \sim C'\Delta x$

- Dirichlet transmission conditions: $|\rho| \approx 1 - \alpha\Delta x$,
- Taylor approximation: $|\rho| \approx 1 - \beta\sqrt{\Delta x}$,
- Optimization: $p \approx C_p \Delta x^{-\frac{1}{5}}$, $q \approx C_q \Delta x^{\frac{3}{5}}$, $|\rho| \approx 1 - O(\Delta x^{\frac{1}{5}})$.

Conclusion for parabolic problems

- Robin transmission conditions are better than Dirichlet, but second order transmission conditions improve significantly.
- overlap is better if possible, but nonoverlapping with second order should be considered if not.
- The convergence rate is almost independent of the discretization parameters.
- Very robust when applied to variable coefficients.

Outline

Hyperbolic equations

Finite speed of propagation $->$ convergence in a finite number of steps.

The 1-D wave equation with discontinuous coefficients

Nonoverlapping scheme. Convergence properties

Optimal convergence with local transmission conditions on time windows.
Convergent finite volumes schemes.
M. Gander, L.H. et F. Nataf,DD11,1998; SINUM 2003.

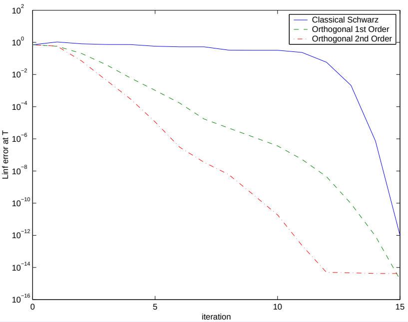
Nonoverlapping scheme. Mesh refinement

Allows for keeping the global order of the scheme (2).
L.H. JCA 2005.

The 2-D wave equation

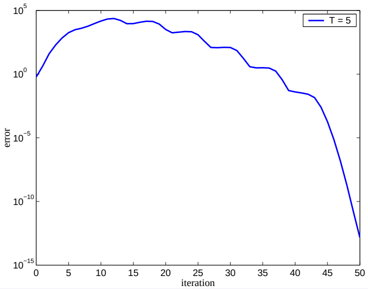
Overlapping Schwarz

Use the second-order absorbing boundary conditions of Engquist-Majda WITH OVERLAP to absorb evanescent waves.
The size of the overlap is optimized such as to absorb the high angle propagation. No strategy without overlap (so far!)
M. Gander et L.H, M. of Comp. 2005.



$$i\partial_t u + \Delta u + V(x)u = 0$$

Classical Schwarz



L.H. et Jérémie Szeftel, arkiv 2006.

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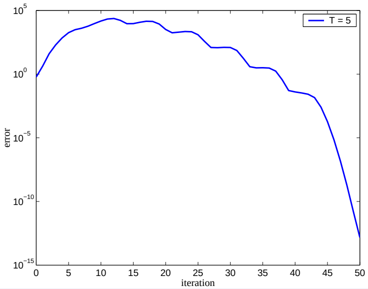
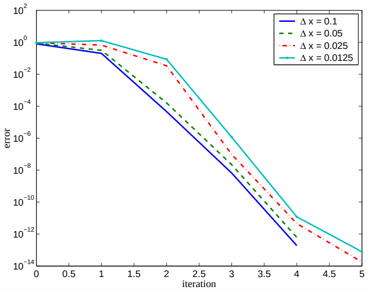


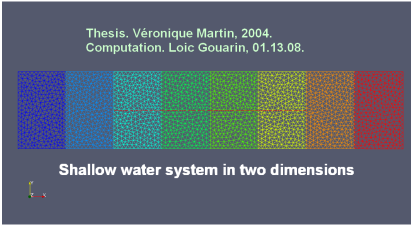
Optimal operator



L.H. et Jérémie Szeftel, arkiv 2006.

Outline

- Theory: nonlinear problems Very good results for the Semilinear wave equation in 1.D.
  L.H. and J. Szeftel, Math of Comp, to appear
- applications to the real world
  - environnement: porous media , see O. Pironneau and C. Japhet in minisymposium.
  - oceanography: primitive equations, inclusion in operational code.



Primitive equations. Ongoing work (Merlet, Audusse, Dreyfuss)

$$\partial_t U_h + U_h \cdot \nabla_h U_h - \nu \Delta U_h + fBU_h + \frac{1}{\rho_0}\nabla_h p = 0,$$

$$\nabla_h \cdot U_h + \partial_z w = 0,$$

$$\partial_z p + \rho_0 g = 0,$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$(U_h, w) = (u, v, w)$, pressure p,ρ density

free boundary height η.
Cinematic free surface condition $\partial_t \zeta + U_h \cdot \nabla_h \zeta - w(\zeta) = 0$,
Equilibrium of surface tensions $\nu \partial_z U_h(\zeta) = 0$,$(p - p_{atm})(\zeta) = 0$.

Temperature T and salinity S transported by advection-diffusion equation.

Adimensionalization + linearization – >

Primitive equations. Ongoing work (Merlet, Audusse, Dreyfuss)

$$\partial_t U_h + U_{0,h} \cdot \nabla_h U_h - \frac{1}{Re}\Delta_h U_h - \frac{1}{Re'}\partial_z^2 U_h + \frac{1}{\varepsilon} B U_h + \frac{1}{Fr^2}\nabla_h \zeta = 0,\ z \in (-1, 0),$$

$$\partial_t \zeta + U_{0,h} \cdot \nabla_h \zeta + \nabla_h \cdot \overline{U}_h = 0, \quad \overline{U}_h := \fint_{-H}^{0} U_h(z)\, dz,$$

$$\partial_z U_h(x, y, 0, t) = \partial_z U_h(x, y, -1, t) = 0.$$

- $\varepsilon = U/(fL)$ Rosby number,
- $Re = UL/\nu$ horizontal Reynolds number,
- $Re' = H^2/L^2 Re$ vertical Reynolds number,
- $Fr = U/\sqrt{gH}$ Froude number.

Fourier series in z and y, Laplace transform in time – > optimal transmission operator.

Primitive equations. Ongoing work (Merlet, Audusse, Dreyfuss)

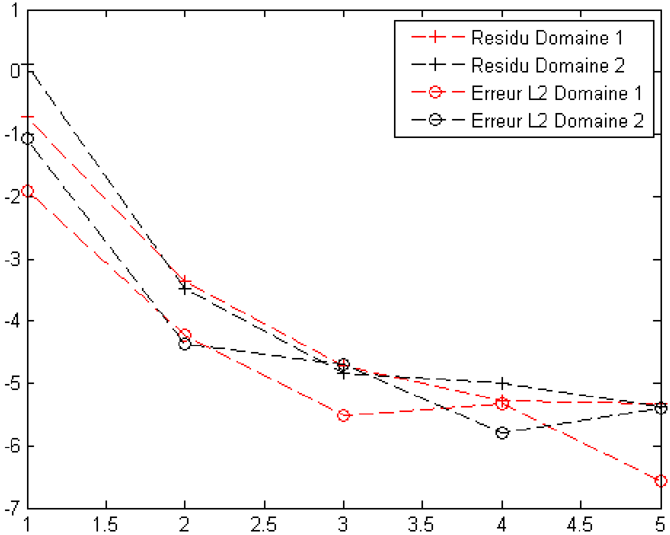
"Robin" transmission operator for the left domain

$$\mathcal{B}_1 X = \begin{pmatrix} \frac{1}{Re}\partial_x u + \left(\frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} - \frac{u_0}{2}\right) u - \frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} v + \frac{1}{2Fr^2 u_0}\overline{u} - \frac{1}{4Fr^2 u_0}\overline{v} \\ \frac{1}{Re}\partial_x v + \left(\frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} - \frac{u_0}{2}\right) v + \frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} u - \frac{1}{4Fr^2 u_0}\overline{u} \end{pmatrix}$$

"Robin" transmission operator for the right domain

$$\mathcal{B}_2 X = \begin{pmatrix} \frac{1}{Re}\partial_x u - \frac{1}{Fr^2}\zeta + \left(-\frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} - \frac{u_0}{2}\right) u + \frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} v - \frac{1}{2Fr^2 u_0}\overline{u} - \frac{1}{4Fr^2 u_0}\overline{v} \\ \frac{1}{Re}\partial_x v + \left(-\frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} - \frac{u_0}{2}\right) v - \frac{\sqrt{2}}{2\sqrt{Re\,\varepsilon}} u - \frac{1}{4Fr^2 u_0}\overline{u} \\ u_0\xi - \overline{u} \end{pmatrix}.$$

Primitive equations. B. Merlet



Collaborators

- Mostly : M. Gander (Université Genève).
- The beginnings, 1D wave equation : F. Nataf (CNRS P6).
- 2D advection-diffusion and Navier-Stokes coupling: P. D'Anfray et J. Ryan (ONERA). V. Martin (Amiens).
- Heterogeneous problems (application to oceanography) : C. Japhet (P13), M. Kern (INRIA), E. Blayo (Grenoble), V. Martin (Amiens), E. Audusse, B. Merlet, P. Dreyfuss (P13) on primitive equations.
- Schrödinger equation and non linear models: J. Szeftel.
- Application to micromagnetism : S. Labbé (U. Grenoble) and K. Santugini(U. Bordeaux)
- construction of **OPTIMISM**, L. Gouarin.

http://www.math.univ-paris13.fr/ halpern