Restricted additive Schwarz method for some inequalities perturbed by a Lipschitz operator

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1 Introduction

The first restricted additive Schwarz methods have been introduced for algebraic linear systems in Cai et al. [1998], Cai and Sarkis [1999] and Frommer and Szyld [2001]. In Frommer et al. [2002] and Nabben and Szyld [2002] the restricted variant of the multiplicative Schwarz method is also analyzed. Numerical experiments have proven that these restricted methods, besides the fact that they sometimes converge faster and also preserve the good properties of the usual additive methods, they reduce the communication time when they are implemented on distributed memory computers. In Efstathiou and Gander [2003], it is explained this fact by showing that even if the restricted method is defined at the matrix level, it can be interpreted as an iteration at the continuous level of the given problem. Restricted additive Schwarz methods for complementarity problems have been introduced in Yang and Li [2012], Zhang et al. [2015], Xu et al. [2014] and Xu et al. [2011].

In the above papers, the methods are approached by a matricial point of view. In this paper, we introduce and analyze a restricted additive method for inequalities perturbed by a Lipschitz operator in the functional framework of the PDEs. Such an approach is not new in the case of the additive and multiplicative Schwarz methods, including the multilevel and multigrid methods for inequalities (see Badea [2008b], Badea [2015] and Badea [2008a], for instance).

In the next section, like in Badea [2008a], we give an existence and uniqueness result concerning the solution of the inequalities we consider¿ Also, we introduce the method as a subspace correction algorithm, prove the convergence and estimate the error in a general framework of a finite dimensional Hilbert space. In Section 3, by introducing the finite element spaces,

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we conclude that both the convergence condition and convergence rate are independent of the mesh parameters, the number of subdomains and of the parameters of the domain decomposition, but the convergence condition is a little more restrictive than the existence and uniqueness condition of the solution.

In a forthcoming paper, by considering the perturbing operator of a particular form, we introduce and analyze some restricted additive Schwarz-Richardson methods for inequalities which do not arise from the minimization of a functional. Also, we shall compare the convergence of these restricted additive methods with the convergence of the corresponding additive methods.

2 Convergence result in a Hilbert space

Let V be finite dimensional real Hilbert space with the basis φ_j, $j = 1, \dots, d$, and let c_d and C_d be two constants such that, for any $v = \sum_{j=1}^d v_j \varphi_j \in V$, we have

$$c_d \sum_{j=1}^d ||v_j \varphi_j||^2 \le ||v||^2 \le C_d \sum_{j=1}^d ||v_j \varphi_j||^2 \tag{1}$$

Also, let $V_1, \dots, V_m$ be some closed subspaces of V and $K \subset V$ be a non empty closed convex set. We consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$ and assume that there exist two real numbers $\alpha,\ \beta > 0$ for which

$$\alpha ||v-u||^2 \le \langle F'(v) - F'(u), v-u \rangle \text{ and } ||F'(v) - F'(u)||_{V'} \le \beta ||v-u|| \tag{2}$$

for any $u, v \in V$. Above, we have denoted by F' the Gâteaux derivative of F. Following the way in Glowinski et al. [1981], we can prove that for any $u, v \in V$, we have

$$\langle F'(u), v-u \rangle + \frac{\alpha}{2} ||v-u||^2 \le F(v) - F(u) \le \langle F'(u), v-u \rangle + \frac{\beta}{2} ||v-u||^2 \tag{3}$$

Also, we consider an operator $T : V \to V'$ with the property that there exists $\gamma > 0$ such that

$$||T(v) - T(u)||_{V'} \le \gamma ||v-u|| \text{ for any } u,\ v \in V. \tag{4}$$

By using the above functional $F : V \to \mathbf{R}$, we also introduce the functional $\mathcal{F} : V \to \mathbf{R}$ defined as $\mathcal{F}(v) = \sum_{j=1}^d F(v_j \varphi_j)$. Evidently, the derivative $\mathcal{F}'$ of $\mathcal{F}$ at $u = \sum_{j=1}^d u_j \varphi_j$ in the direction $v = \sum_{j=1}^d v_j \varphi_j$ is written as $\langle \mathcal{F}'(u), v \rangle = \sum_{j=1}^d \langle F'(u_j \varphi_j), v_j \varphi_j \rangle$ and, in view of (3), we have

$$\begin{aligned} &\langle \mathcal{F}'(u), v-u \rangle + \tfrac{\alpha}{2} \sum_{j=1}^d ||(v_j - u_j)\varphi_j||^2 \le \mathcal{F}(v) - \mathcal{F}(u) \\ &\le \langle \mathcal{F}'(u), v-u \rangle + \tfrac{\beta}{2} \sum_{j=1}^d ||(v_j - u_j)\varphi_j||^2 \end{aligned} \tag{5}$$

for any $u = \sum_{j=1}^{d} u_j\varphi_j,\ v = \sum_{j=1}^{d} v_j\varphi_j \in V$. Evidently, from the convexity of F we get that $\mathcal{F}$ is also a convex functional. Finally, we assume that if K is not bounded then the functional $\mathcal{F}$ is coercive in the sense that $\mathcal{F}(v)/||v|| \to \infty$ as $||v|| \to \infty,\ v \in V$.

Now, we define an operation $* : V \times V \to V$ by

$$u * v = \sum_{j=1}^{d} u_j v_j \varphi_j \text{ for any } u = \sum_{j=1}^{d} u_j\varphi_j \text{ and } v = \sum_{j=1}^{d} v_j\varphi_j \in V \quad (6)$$

We fix some functions $\theta_i = \sum_{j=1}^{d} \theta_{ij}\varphi_j \in V_i,\ i = 1, \dots, m$, and assume that they have the property

$$0 \le \theta_{ij} \le 1 \text{ and } \sum_{i=1}^{m} \theta_{ij} = 1 \text{ for any } j = 1, \dots, m \quad (7)$$

i.e., in some sense, they supply a unity decomposition associated with the subspaces $V_1, \dots, V_m$. Also, we assume that the convex set K has the property

Property 1. If $v, w \in K$ and $\theta = \sum_{j=1}^{d} \theta_j\varphi_j \in V$ with $0 \le \theta_j \le 1, j = 1, \dots, d$, then $\theta * v + (\bar{1} - \theta) * w \in K$.

Above and in what follows in this section, $\sum_{j=1}^{d} \varphi_j$ is denoted by $\bar{1}$. Using (6), we have $\bar{1} * v = v$ for any $v \in V$. Finally, we consider the problem

$$u \in K : \langle \mathcal{F}'(u), v - u\rangle - \langle T(u), v - u\rangle \ge 0, \text{ for any } v \in K. \quad (8)$$

which is a variational inequality perturbed by the operator T. Concerning the existence and the uniqueness of the solution of this problem we have the following result (see Badea [2008a], for the proof of a similar result).

Proposition 1. *If $\frac{\gamma}{\alpha} C_d < 1$, then problem (8) has a unique solution.*

Since the functional $\mathcal{F}$ is convex and differentiable, problem (8) is equivalent with the minimization problem

$$u \in K : \mathcal{F}(u) - \langle T(u), u\rangle \le \mathcal{F}(v) - \langle T(u), v\rangle, \text{ for any } v \in K. \quad (9)$$

We write the restricted additive algorithm for the solution of problem (8) as

Algorithm 1 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we solve the inequalities: find $w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K$ such that*

$$\begin{aligned} &\langle \mathcal{F}'(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^n), v_i - w_i^{n+1}\rangle \ge 0, \\ &\textit{for any } v_i \in V_i,\ u^n + v_i \in K, \end{aligned} \quad (10)$$

for $i = 1, \dots, m$, and then we update $u^{n+1} = u^n + \sum_{i=1}^{m} \theta_i * w_i^{n+1}$.

Now we prove

Theorem 1. *Let u be the solution of problem (8), and u^n, $n \ge 1$, be its approximations obtained from Algorithm 1. If $\frac{\gamma}{\alpha} C_d \le \vartheta_{\max}$, where $\vartheta_{\max}$ is* *defined in (27), then Algorithm 1 is convergent for any initial guess* $u^0 \in K$ *and the error estimates*

$$\begin{array}{l} \mathcal{F}(u^n) - \langle T(u), u^n\rangle - \mathcal{F}(u) + \langle T(u), u\rangle \\ \le \left(\frac{\tilde{C}}{\tilde{C}+1}\right)^n \left[\mathcal{F}(u^0) - \langle T(u), u^0\rangle - \mathcal{F}(u) + \langle T(u), u\rangle\right] \end{array} \quad (11)$$

and

$$\begin{array}{l} \sum_{j=1}^d ||(u_j^n - u_j)\varphi_j||^2 \le \frac{2}{\alpha}\left(\frac{\tilde{C}}{\tilde{C}+1}\right)^n \left[\mathcal{F}(u^0) - \langle T(u), u^0\rangle \right. \\ \left. -\mathcal{F}(u) + \langle T(u), u\rangle\right] \end{array} \quad (12)$$

hold for any $n \ge 1$*, where constant* $\tilde{C}$ *is given in (28).*

Proof. Using (5), (7) and (10), we get

$$\begin{array}{l} \mathcal{F}(u^{n+1}) - \mathcal{F}(u) + \langle T(u), u - u^{n+1}\rangle + \frac{\alpha}{2}\sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 \\ \le \langle \mathcal{F}'(u^{n+1}), u^{n+1} - u\rangle + \langle T(u), u - u^{n+1}\rangle \\ \le \sum_{i=1}^m \langle \mathcal{F}'(u^n + w_i^{n+1}) - \mathcal{F}'(u^{n+1}), \theta_i * (u - u^n) + (\bar{1} - \theta_i) * w_i^{n+1} - w_i^{n+1}\rangle \\ - \sum_{i=1}^m \langle T(u^n), \theta_i * (u - u^n) + (\bar{1} - \theta_i) * w_i^{n+1} - w_i^{n+1}\rangle + \langle T(u), u - u^{n+1}\rangle \end{array}$$

Above, we have used the fact that $\theta_i*(u-u^n)+(\bar{1}-\theta_i)*w_i^{n+1} \in V_i$ and, in view of Property 1, $u^n+\theta_i*(u-u^n)+(\bar{1}-\theta_i)*w_i^{n+1} = (\bar{1}-\theta_i)*(u^n+w_i^{n+1})+\theta_i*u \in K$ and therefore, we can replace v_i by $\theta_i * (u - u^n) + (\bar{1} - \theta_i) * w_i^{n+1}$ in (10). Consequently, we have

$$\begin{array}{l} \mathcal{F}(u^{n+1}) - \mathcal{F}(u) - \langle T(u), u^{n+1} - u\rangle + \frac{\alpha}{2}\sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 \\ \le \sum_{i=1}^m \langle \mathcal{F}'(u^n + w_i^{n+1}) - \mathcal{F}'(u^{n+1}), \theta_i * (u - u^n - w_i^{n+1})\rangle \\ + \sum_{i=1}^m \langle T(u) - T(u^n), \theta_i * (u - u^n - w_i^{n+1})\rangle \end{array} \quad (13)$$

In view of (2) and (7), we have

$$\begin{array}{l} \sum_{i=1}^m \langle \mathcal{F}'(u^n + w_i^{n+1}) - \mathcal{F}'(u^{n+1}), \theta_i * (u - u^n - w_i^{n+1})\rangle \\ \le \beta \sum_{i=1}^m \sum_{j=1}^d \theta_{ij} ||((1-\theta_{ij})w_{ij}^{n+1} - \sum_{k=1,\, k\ne i}^m \theta_{kj} w_{kj}^{n+1})\varphi_j|| \\ \cdot ||(u_j - u_j^{n+1} - (1-\theta_{ij})w_{ij}^{n+1} + \sum_{k=1,\, k\ne i}^m \theta_{kj} w_{kj}^{n+1})\varphi_j|| \\ \le \beta \sum_{i=1}^m \sum_{j=1}^d \theta_{ij}\left((1-\theta_{ij})||w_{ij}^{n+1}\varphi_j|| + \sum_{k=1,\, k\ne i}^m \theta_{kj}||w_{kj}^{n+1}\varphi_j||\right) \\ \cdot\left(||(u_j - u_j^{n+1})\varphi_j|| + (1-\theta_{ij})||w_{ij}^{n+1}\varphi_j|| + \sum_{k=1,\, k\ne i}^m \theta_{kj}||w_{kj}^{n+1}\varphi_j||\right) \\ \le \beta \sum_{i=1}^m \sum_{j=1}^d \theta_{ij}\left[(1 + \frac{1}{2\varepsilon_1})\left((1-\theta_{ij})||w_{ij}^{n+1}\varphi_j||\right.\right. \\ \left.\left. + \sum_{k=1,\, k\ne i}^m \theta_{kj}||w_{kj}^{n+1}\varphi_j||\right)^2 + \frac{\varepsilon_1}{2}||(u_j - u_j^{n+1})\varphi_j||^2\right] \le 2\beta(1 + \frac{1}{2\varepsilon_1}) \\ \cdot \sum_{i=1}^m \sum_{j=1}^d \theta_{ij}(1-\theta_{ij})\left((1-\theta_{ij})||w_{ij}^{n+1}\varphi_j||^2 + \sum_{k=1,\, k\ne i}^m \theta_{kj}||w_{kj}^{n+1}\varphi_j||^2\right) \\ +\beta\frac{\varepsilon_1}{2}\sum_{j=1}^d ||(u_j - u_j^{n+1})\varphi_j||^2 = 2\beta(1 + \frac{1}{2\varepsilon_1})\sum_{i=1}^m \sum_{j=1}^d \theta_{ij}(1-\theta_{ij}) \\ \cdot(1 - 2\theta_{ij})||w_{ij}^{n+1}\varphi_j||^2 + 2\beta(1 + \frac{1}{2\varepsilon_1})\sum_{i=1}^m \sum_{j=1}^d \theta_{ij}(1-\theta_{ij}) \\ \cdot \sum_{k=1}^m \theta_{kj}||w_{kj}^{n+1}\varphi_j||^2 + \beta\frac{\varepsilon_1}{2}\sum_{j=1}^d ||(u_j - u_j^{n+1})\varphi_j||^2 \end{array}$$

or

$$\begin{array}{l} \sum_{i=1}^m \langle \mathcal{F}'(u^n + w_i^{n+1}) - 4\beta(1 \\ + \frac{1}{2\varepsilon_1})\sum_{i=1}^m \sum_{j=1}^d \theta_{ij}||w_{ij}^{n+1}\varphi_j||^2 + \beta\frac{\varepsilon_1}{2}\sum_{j=1}^d ||(u_j - u_j^{n+1})\varphi_j||^2 \end{array} \quad (14)$$

for any $\varepsilon_1 > 0$. Also, from (4) and (1), we get

$$\sum_{i=1}^m \langle T(u) - T(u^n), \theta_i * (u - u^n - w_i^{n+1})\rangle = \langle T(u) - T(u^n), u - u^n - \sum_{i=1}^m \theta_i * w_i^{n+1}\rangle = \langle T(u) - T(u^n), u - u^{n+1})\rangle \leq \gamma ||u - u^n|| \, ||u - u^{n+1}|| \leq \gamma \left(||u - u^{n+1}|| + || \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} w_{ij}^{n+1} \varphi_j || \right) ||u - u^{n+1}|| \leq \gamma C_d \left((1 + \frac{\varepsilon_2}{2}) \sum_{j=1}^d ||(u_j - u_j^{n+1})\varphi_j||^2 + \frac{1}{2\varepsilon_2} \sum_{j=1}^d || \sum_{i=1}^m \theta_{ij} w_{ij}^{n+1} \varphi_j ||^2 \right)$$

i.e., using (7), we have

$$\sum_{i=1}^m \langle T(u) - T(u^n), \theta_i * (u - u^n - w_i^{n+1})\rangle \leq \gamma C_d \left((1 + \frac{\varepsilon_2}{2}) \cdot \sum_{j=1}^d ||(u_j - u_j^{n+1})\varphi_j||^2 + \frac{1}{2\varepsilon_2} \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} ||w_{ij}^{n+1} \varphi_j||^2 \right) \tag{15}$$

for any $\varepsilon_2 > 0$. From (13), (14) and (15), we get

$$\mathcal{F}(u^{n+1}) - \mathcal{F}(u) - \langle T(u), u^{n+1} - u\rangle + \left(\frac{\alpha}{2} - \beta \frac{\varepsilon_1}{2} - \gamma C_d (1 + \frac{\varepsilon_2}{2}) \right) \cdot \sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 \leq \left[4\beta (1 + \frac{1}{2\varepsilon_1}) + \gamma C_d \frac{1}{2\varepsilon_2} \right] \cdot \sum_{i=1}^m \sum_{j=1}^d \theta_{ij} ||w_{ij}^{n+1} \varphi_j||^2 \tag{16}$$

for any $\varepsilon_1, \ \varepsilon_2 > 0$.

Now, by taking $v_i = (\bar{1} - \theta_i) * w_i^{n+1}$ in (10), for $i = 1, \ldots, m$, we get

$$\sum_{j=1}^d \theta_{ij} \left[\langle F'((u_j^n + w_{ij}^{n+1})\varphi_j), -w_{ij}^{n+1} \varphi_j \rangle - \langle T(u^n), -w_{ij}^{n+1} \varphi_j \rangle \right] \geq 0 \tag{17}$$

In view of (7), the convexity of F, (2) and the above equation, we have

$$\mathcal{F}(u^{n+1}) - \mathcal{F}(u^n) \leq \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} [F((u_j^n + w_{ij}^{n+1})\varphi_j) - F(u_j^n \varphi_j)] \leq \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} \left[-\frac{\alpha}{2} ||w_{ij}^{n+1}||^2 - \langle F'((u_j^n + w_{ij}^{n+1})\varphi_j), -w_{ij}^{n+1} \varphi_j \rangle \right] = \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} \left[-\frac{\alpha}{2} ||w_{ij}^{n+1} \varphi_j||^2 - \langle T(u^n), -w_{ij}^{n+1} \varphi_j \rangle - \langle F'((u_j^n + w_{ij}^{n+1})\varphi_j), -w_{ij}^{n+1} \varphi_j \rangle + \langle T(u^n), -w_{ij}^{n+1} \varphi_j \rangle \right] \leq \sum_{j=1}^d \sum_{i=1}^m \theta_{ij} \left[-\frac{\alpha}{2} ||w_{ij}^{n+1} \varphi_j||^2 - \langle T(u^n), -w_{ij}^{n+1} \varphi_j \rangle \right]$$

Consequently, we have

$$\frac{\alpha}{2} \sum_{i=1}^m \sum_{j=1}^d \theta_{ij} ||w_{ij}^{n+1} \varphi_j||^2 \leq \mathcal{F}(u^n) - \mathcal{F}(u^{n+1}) + \langle T(u), u^{n+1} - u^n \rangle + \langle T(u^n) - T(u), u^{n+1} - u^n \rangle \tag{18}$$

With a proof similar to that of (15), we get

$$\langle T(u^n) - T(u), u^{n+1} - u^n \rangle \leq \gamma C_d \left[\frac{\varepsilon_3}{2} \sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 + (1 + \frac{1}{2\varepsilon_3}) \sum_{i=1}^m \sum_{j=1}^d \theta_{ij} ||w_{ij}^{n+1} \varphi_j||^2 \right] \tag{19}$$

for any $\varepsilon_3 > 0$.

Consequently, from (18) and (19), we get

$$\left[\frac{\alpha}{2} - \gamma C_d (1 + \frac{1}{2\varepsilon_3}) \right] \sum_{i=1}^m \sum_{j=1}^d \theta_{ij} ||w_{ij}^{n+1} \varphi_j||^2 \leq \mathcal{F}(u^n) - \mathcal{F}(u^{n+1}) + \langle T(u), u^{n+1} - u^n \rangle + \gamma C_d \frac{\varepsilon_3}{2} \sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 \tag{20}$$

for any $\varepsilon_3 > 0$. Let us write

$$C_1 = \frac{\alpha}{2} - \gamma C_d (1 + \frac{1}{2\varepsilon_3}) \tag{21}$$

For values of γ, α and ε_3 such that $C_1 > 0$, from (16) and (20), we have

$$\begin{aligned} &\mathcal{F}(u^{n+1}) - \mathcal{F}(u) - \langle T(u), u^{n+1} - u\rangle + C_2 \sum_{j=1}^d ||(u_j^{n+1} - u_j)\varphi_j||^2 \\ &\leq \tilde{C}\left[\mathcal{F}(u^n) - \mathcal{F}(u^{n+1}) + \langle T(u), u^{n+1} - u^n\rangle\right] \end{aligned} \tag{22}$$

where

$$\tilde{C} = \frac{1}{C_1}\left(4\beta(1 + \frac{1}{2\varepsilon_1}) + \gamma C_d \frac{1}{2\varepsilon_2}\right) \tag{23}$$

and

$$C_2 = \frac{\alpha}{2} - \beta\frac{\varepsilon_1}{2} - \gamma C_d(1 + \frac{\varepsilon_2}{2}) - \gamma C_d \frac{\varepsilon_3}{2}\tilde{C} \tag{24}$$

In view of (22), assuming that $C_2 \geq 0$, we easily get (11). Estimation (12) follows from (11) and (3) and (8). Indeed, we have

$$\begin{aligned} &\mathcal{F}(u^n) - \mathcal{F}(u) - \langle T(u), u^n - u\rangle = \sum_{j=1}^d F(u_j^n\varphi_j) - \sum_{j=1}^d F(u_j\varphi_j) \\ &-\langle T(u), u^n - u\rangle \geq \sum_{j=1}^d \langle F'(u_j\varphi_j), (u_j^n - u_j)\varphi_j\rangle \\ &+\frac{\alpha}{2}\sum_{j=1}^d ||(u_j^n - u_j)\varphi_j||^2 - \langle T(u), u^n - u\rangle = \langle \mathcal{F}'(u), u^n - u\rangle \\ &-\langle T(u), u^n - u\rangle + \frac{\alpha}{2}\sum_{j=1}^d ||(u_j^n - u_j)\varphi_j||^2 \geq \frac{\alpha}{2}\sum_{j=1}^d ||(u_j^n - u_j)\varphi_j||^2 \end{aligned} \tag{25}$$

Using (23), (24) and (21), condition $C_2 \geq 0$ can be written as $C_2 = A - 4B\beta - \frac{\beta}{2}(\varepsilon_1 + 4\frac{B}{\varepsilon_1}) - \frac{\gamma C_d}{2}(\varepsilon_2 + \frac{B}{\varepsilon_2}) \geq 0$ with $A = \frac{\alpha}{2} - \gamma C_d$ and $B = \frac{\gamma C_d \frac{\varepsilon_3}{2}}{A - \gamma C_d \frac{1}{2\varepsilon_3}}$
The maximum value of C_2 is obtained for

$$\varepsilon_1 = 2\frac{\gamma C_d}{A} \quad \varepsilon_2 = \varepsilon_3 = \frac{\gamma C_d}{A} \tag{26}$$

Consequently, for these values, we should have
$C_{2\max} = \frac{\alpha^3}{A^2}\left[\frac{1}{2}(\frac{1}{2} - \frac{\gamma C_d}{\alpha})(\frac{1}{2} - 2\frac{\gamma C_d}{\alpha}) - \frac{2\beta}{\alpha}\frac{\gamma C_d}{\alpha}(\frac{1}{2} + \frac{\gamma C_d}{\alpha})\right] \geq 0$, or

$$C_d\frac{\gamma}{\alpha} \leq \frac{1}{\sqrt{16\frac{\beta^2}{\alpha^2} + 40\frac{\beta}{\alpha} + 1} + 4\frac{\beta}{\alpha} + 3} = \vartheta\max \tag{27}$$

By a simple calculus, we see that if (27) holds, then condition $C_1 > 0$ is satisfied for the value of ε_3 in (26). Finally, by replacing ε_1, ε_2 and ε_3 in (23) with their values in (26), we get

$$\tilde{C} = 1 + \frac{2\beta}{\alpha}\frac{6\frac{\gamma C_d}{\alpha} + 1}{\frac{\gamma C_d}{\alpha}\left(1 - 2\frac{\gamma C_d}{\alpha}\right)} \geq 1 + \frac{2\beta}{\alpha}\frac{6\vartheta\max + 1}{\vartheta\max\,(1 - 2\vartheta\max)} \tag{28}$$

It should be noted that the convergence condition and the convergence rate are independent of the number m of subspaces.

3 Restricted additive Schwarz method in a finite element space

Let Ω be an open bounded domain in $\mathbf{R}^N$, $N = 1, 2$ or 3, and we consider a simplicial regular mesh partition $\mathcal{T}_h$. We assume that domain Ω is decomposed in m subdomains, $\Omega = \bigcup_{i=1}^m \Omega_i$, and that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. We associate to the mesh partition $\mathcal{T}_h$ the piecewise linear finite element space $V_h \subset H_0^1(\Omega)$ and to the domain decomposition the subspaces $V_h^i \subset H_0^1(\Omega_i)$. We assume that the convex set $K_h \subset V_h$ has the following

Property 2. If $v, w \in K_h$, and if $\theta \in V_h$, $0 \leq \theta \leq 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.

Above and also in the following, we denote by L_h the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$. It is easy to see that the convex sets of two-obstacle type have Property 2.

Now, we estimate C_d in (1). Given a triangle $\tau \in \mathcal{T}_h$, let $J_\tau = \{1 \leq j \leq d : \tau \subset \text{ supp } \varphi_j\}$. Then, for a $v = \sum_{j=1}^d v_j \varphi_j \in V_h$, and using the norm of $H^1(\Omega)$ we have
$||v||^2 = \sum_\tau ||v||_\tau^2 = \sum_\tau \left(\sum_{j \in J_\tau} v_j \varphi_j, \sum_{j \in J_\tau} v_j \varphi_j \right)_\tau \leq$
$\sum_\tau |J_\tau| \sum_{j \in J_\tau} ||v_j \varphi_j||_\tau^2 \leq \sum_\tau |J_\tau| \sum_{j=1}^d ||v_j \varphi_j||_\tau^2 \leq C_d \sum_{j=1}^d \sum_\tau ||v_j \varphi_j||_\tau^2$
$= C_d \sum_{j=1}^d ||v_j \varphi_j||^2$
where we have denoted $C_d = \max_{\tau \in \mathcal{T}_h} |J_\tau|$. Since $\mathcal{T}_h$ are simplicial meshes, then $\max_\tau |J_\tau|$ is independent of the mesh parameters when $h \to 0$. Therefore, we can consider that C_d *is independent of the domain or mesh parameters*.

Finally, it is evident that $*$ in (6) can be written as $u * v = L_h(uv)$ for any u, $v \in V_h$. Moreover, if $\{\theta_1, \ldots, \theta_m\} \subset V_h$ is a unity partition associated with the domain decomposition, then (7) holds for any $v \in V_h$. Besides that, in view of Property 2 of the convex set K_h, this convex set also has Property 1. In the matricial description of the method, some restriction operators, $R_1^0, \ldots, R_m^0$, are used instead of our unity partition $\{\theta_1, \ldots, \theta_m\}$. If we associate to a $v = \sum_{j=1}^d v_j \varphi_j \in V_h$ the vector $(v_1, \ldots, v_d)$ then $\theta_i * v$ is associated with $R_i^0(v_1, \ldots, v_d)$. In general, these restriction operators supply a minimum overlap i.e., with our notations, the components θ_{ij} of the functions $\theta_i = \sum_{j=0}^m \theta_{ij} \varphi_j$ satisfy either $\theta_{ij} = 1$ or $\theta_{ij} = 0$. A PDEs definition of the method using a unity partition associated to the domain decomposition and which is very close to that introduced by us is given in Dolean et al. [2015].

From (27), (28) and the above comments we can conclude that the convergence condition and convergence rate of Algorithm 1 are independent of the mesh parameters and of both the number of subdomains and the parameters of the domain decomposition, but the convergence condition is more restrictive than the existence and uniqueness condition of the solution given in Proposition 1.

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