A Smoother Based on Nonoverlapping Domain Decomposition Methods for $H(\mathrm{div})$ Problems: A Numerical Study

Susanne C. Brenner and Duk-Soon Oh

Abstract The purpose of this paper is to introduce a V-cycle multigrid method for vector field problems discretized by the lowest order Raviart-Thomas hexahedral element. Our method is connected with a smoother based on a nonoverlapping domain decomposition method. We present numerical experiments to show the effectiveness of our method.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ and $H_0(\mathrm{div};\Omega)$ be the space of square integrable vector fields on Ω that have square integrable divergence in Ω and vanishing normal components on $\partial\Omega$ (cf. [7]). In this paper we consider a multigrid method for the following problem: Find $\boldsymbol{u} \in H_0(\mathrm{div};\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall\,\boldsymbol{v} \in H_0(\mathrm{div};\Omega), \tag{1}$$

where

$$a(\boldsymbol{w},\boldsymbol{v}) = \alpha(\mathrm{div}\,\boldsymbol{w},\mathrm{div}\,\boldsymbol{v}) + \beta(\boldsymbol{w},\boldsymbol{v}), \tag{2}$$

and $(\cdot,\cdot)$ is the inner product on $L_2(\Omega)$ (or $[L_2(\Omega)]^3$). We assume that $\boldsymbol{f} \in [L_2(\Omega)]^3$ and α and β are positive. Unlike the scalar elliptic equation case, multigrid methods for the problem (1) with simple smoothers do not work. We need a special treatment

Susanne C. Brenner
Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA
e-mail: brenner@math.lsu.edu

Duk-Soon Oh
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
e-mail: duksoonoh@gmail.com, duksoon@math.rutgers.edu

for the smoother. In [2–4, 9], an overlapping domain decomposition preconditioner was employed in the construction of the smoother.

Our goal is to develop multigrid methods in the same spirit but using nonoverlapping domain decomposition preconditioners instead, which reduce the dimensions of the subproblems that have to be solved. We note that other multigrid methods for $H(\text{div})$ were investigated in [8, 10].

Applications of fast solvers for $H(\text{div})$ problems are discussed for example in [2, 11–13, 16]. In particular the multigrid method in this paper can be applied to a mixed method for second order partial differential equations based on a first-order system least-squares formulation [2, 6], which is equivalent to our model problem. It can also be used as an effective preconditioner for $H(\text{div})$ problems with variable coefficients. The model problem also arise in Reissner-Mindlin plates [1] and Brinkman equations [15].

In [5], there are similar ingredients and convergence analysis for the convex domain and the constant coefficient case. In this paper, we mainly focus on the numerical study that is not covered by the theory in [5].

The rest of this paper is organized as follows. We present the standard discretization of (1) by the lowest order Raviart-Thomas hexahedral element in Section 2. We next introduce the V-cycle multigrid method in Section 3. Finally, numerical experiments are presented in Section 4.

2 The Discrete Problem

Let $\mathscr{T}_h$ be a hexahedral triangulation of Ω. The lowest order Raviart-Thomas $H(\text{div})$ conforming finite element space [14] is denoted by V_h. A vector field $\boldsymbol{v}$ belongs to V_h if and only if it belongs to $H_0(\text{div};\Omega)$ and takes the form

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 x_1 \\ b_2 x_2 \\ b_3 x_3 \end{bmatrix}$$

on each hexahedral element, where the a_i's and b_i's are constants. On each hexahedral element T the vector field $\boldsymbol{v}$ is determined by the six degrees of freedom defined by the average of the normal component on each face. The discrete problem for (1) is to find $\boldsymbol{u}_h \in V_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \qquad \forall\, \boldsymbol{v} \in V_h. \tag{3}$$

In the multigrid approach we solve (3) on a sequence of triangulations $\mathscr{T}_0, \mathscr{T}_1, \ldots$, where $\mathscr{T}_0$ is an initial triangulation of Ω by hexahedral elements and $\mathscr{T}_k$ $(k \geq 1)$ is obtained from $\mathscr{T}_{k-1}$ by uniform subdivision. We will denote the lowest order Raviart-Thomas finite element space associated with $\mathscr{T}_k$ by V_k. The k-th level discrete problem is to find $\boldsymbol{u}_k \in V_k$ such that

$$a(\boldsymbol{u}_k,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_k.$$

Let $A_k : V_k \longrightarrow V_k'$ be defined by

$$\langle A_k\boldsymbol{w},\boldsymbol{v}\rangle = a(\boldsymbol{w},\boldsymbol{v}) \qquad \forall \boldsymbol{v},\boldsymbol{w} \in V_k, \tag{4}$$

where $\langle\cdot,\cdot,\rangle$ is the canonical bilinear form on $V_k' \times V_k$. We can then rewrite the k-th level discrete problem as

$$A_k\boldsymbol{u}_k = f_k, \tag{5}$$

where $f_k \in V_k'$ is defined by

$$\langle f_k,\boldsymbol{v}\rangle = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_k.$$

Multigrid methods are optimal order iterative methods for equations of the form

$$A_k\boldsymbol{z} = g \tag{6}$$

that includes (5) as a special case.

3 A V-Cycle Multigrid Method

Since the finite element spaces are nested, we can take the coarse-to-fine operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the natural injection. The fine-to-coarse operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ is then defined by

$$\langle I_k^{k-1}\ell,\boldsymbol{v}\rangle = \langle \ell, I_{k-1}^k\boldsymbol{v}\rangle \qquad \forall \ell \in V_k',\, \boldsymbol{v} \in V_{k-1}. \tag{7}$$

We will use a smoother of the form

$$z_{\text{new}} = z_{\text{old}} + M_k^{-1}(g - A_k z_{\text{old}}) \tag{8}$$

for the equation (6), where $M_k^{-1} : V_k' \longrightarrow V_k$ is a nonoverlapping domain decomposition preconditioner defined below.

3.1 A Nonoverlapping Domain Decomposition Preconditioner

To conform with standard terminology in domain decomposition, in this subsection we will denote $\mathscr{T}_{k-1}$ by $\mathscr{T}_H$ and $\mathscr{T}_k$ by $\mathscr{T}_h$. (Thus each element in $\mathscr{T}_H$ is partitioned into eight elements in $\mathscr{T}_h$). The spaces V_{k-1} and V_k are denoted by V_H and V_h respectively. The preconditioner M_k^{-1} in (8) is denoted by M_h^{-1} here. It is constructed by substructuring.

For each element $T \in \mathscr{T}_H$, we define the twelve dimensional subspace V_h^T of V_h by

$$V_h^T = \{\boldsymbol{v} \in V_h : \boldsymbol{v} = 0 \text{ on } \Omega \setminus T\}. \tag{9}$$

The natural injection from V_h^T into V_h is denoted by J_T and the operator $A_T : V_h^T \longrightarrow (V_h^T)'$ is defined by

$$\langle A_T \boldsymbol{w}, \boldsymbol{v} \rangle = a(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in V_h^T. \tag{10}$$

Let $\mathscr{F}_H$ be the set of the interior faces of the triangulation $\mathscr{T}_H$. Given any $F \in \mathscr{F}_H$ that is the common face of two elements T_F^+ and T_F^- in $\mathscr{T}_H$, we define the four dimensional subspace V_h^F of V_h by

$$V_h^F = \{\boldsymbol{v} \in V_h : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Omega \setminus (T_F^- \cup T_F^+) \quad \text{and} \quad a(\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall \boldsymbol{w} \in (V_h^{T_F^-} + V_h^{T_F^+})\}. \tag{11}$$

The natural injection from V_h^F into V_h is denoted by J_F and the operator $A_F : V_h^F \longrightarrow (V_h^F)'$ is defined by

$$\langle A_F \boldsymbol{w}, \boldsymbol{v} \rangle = a(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in V_h^F. \tag{12}$$

if $\boldsymbol{w} \in V_h$ has the same degrees of freedom as $\boldsymbol{v}$ on $\partial T_F^+ \cup \partial T_F^-$.

The subspaces associated with the elements and interior faces of $\mathscr{T}_H$ form a direct sum decomposition of V_h:

$$V_h = \sum_{T \in \mathscr{T}_H} V_h^T + \sum_{F \in \mathscr{F}_H} V_h^F, \tag{13}$$

and the preconditioner M_h^{-1} is given by

$$M_h^{-1} = \eta_F \Big(\sum_{T \in \mathscr{T}_H} J_T A_T^{-1} J_T^t + \sum_{F \in \mathscr{F}_H} J_F A_F^{-1} J_F^t \Big), \tag{14}$$

where η_F is a damping factor and $J_T^t : V_h' \longrightarrow (V_h^T)'$ (resp. $J_F^t : V_h' \longrightarrow (V_h^F)'$) is the transpose of J_T (resp. J_F) with respect to the canonical bilinear forms.

3.2 The k^{th} Level V-Cycle Multigrid Algorithm

The output $MG(k, g, z_0, m)$ of the k^{th} level (symmetric) multigrid V-cycle algorithm for (6), with initial guess $z_0 \in V_k$ and m smoothing steps, is defined by the following recursive steps:

For $k = 0$, the output is obtained from a direct method:

$$MG(0, g, z_0, m) = A_0^{-1} g.$$

For $k \geq 1$, we set

$$
\begin{aligned}
z_l &= z_{l-1} + M_k^{-1}\left(g - A_k z_{l-1}\right) \qquad \text{for } 1 \le l \le m,\\
\overline{g} &= I_k^{k-1}\left(g - A_k z_m\right),\\
z_{m+1} &= z_m + I_{k-1}^k MG\left(k-1, \overline{g}, 0, m\right),\\
z_l &= z_{l-1} + M_k^{-1}\left(g - A_k z_{l-1}\right) \qquad \text{for } m+2 \le l \le 2m+1.
\end{aligned}
$$

The output of $MG(k, g, z_0, m)$ is z_{2m+1}.

Remark 1. Given $\ell \in V_k'$, the cost of computing $M_k^{-1}\ell$ is $O(n_k)$, where n_k is the dimension of V_k. Therefore the overall cost for computing $MG(k, g, z_0, m)$ is also $O(n_k)$.

If the domain Ω is convex, we have the following convergence theorem:

Theorem 1. *If $z \in V_k$ and $g \in V_k'$ satisfy $A_k z = g$, then we have*

$$
\|z - MG(k, g, z_0, m)\|_a \le \frac{C}{C+2m}\|z - z_0\|_a \qquad \forall k \ge 1,
$$

where $\|\cdot\|_a^2 = a(\cdot,\cdot)$.

Due to space restriction, a detailed analysis will not be reported here. Further details are provided in [5].

4 Numerical Results

4.1 Jump Coefficient

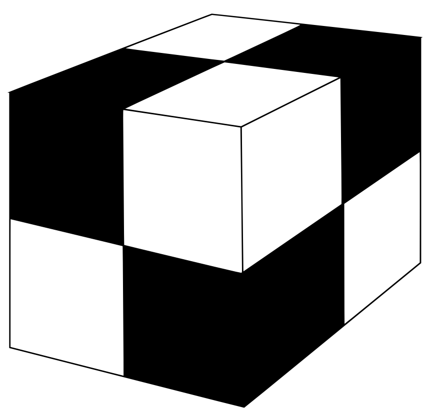

Fig. 1: Checkerboard distribution of the coefficients

In the first experiment we consider (1) on the unit cube $\Omega = (0,1)^3$. We apply multigrid algorithms with smoothers introduced in Section 3.1. The damping factor η_F is taken to be $1/11$. The initial triangulation $\mathscr{T}_0$ consists of eight identical cubes and we use the coefficients α and β that have jumps across the interface between the sub-cubes with a checkerboard pattern as in Fig. 1. We estimate the contraction numbers of the k^{th} level V-cycle multigrid method for $k = 1, \dots, 5$ and for m smoothing steps, where $m = 1, \dots, 6$. We report the contraction numbers obtained by computing the largest eigenvalue of the error propagation operators. The results are presented in Table 1. The uniform convergence of the V-cycle multigrid methods for $m \geq 1$ is clearly observed and the method is not sensitive to the jumps of coefficients.

Table 1: Contraction numbers of the V-cycle multigrid method for the unit cube. α_b and β_b for the black subregions and α_w and β_w for the white subregions as indicated in a checkerboard pattern as in Fig. 1

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
	$\alpha_b = 0.01, \beta_b = 100, \alpha_w = 1, \beta_w = 1$					
$k=1$	8.3e-1	6.8e-1	4.7e-1	2.2e-1	5.1e-2	4.7e-3
$k=2$	9.0e-1	8.2e-1	7.1e-1	5.1e-1	3.2e-1	2.7e-1
$k=3$	9.3e-1	8.8e-1	7.9e-1	6.4e-1	5.2e-1	4.7e-1
$k=4$	9.3e-1	9.0e-1	8.4e-1	7.2e-1	6.4e-1	6.0e-1
$k=5$	9.3e-1	9.0e-1	8.6e-1	7.8e-1	6.9e-1	6.9e-1
	$\alpha_b = 0.1, \beta_b = 10, \alpha_w = 1, \beta_w = 1$					
$k=1$	8.7e-1	7.7e-1	6.0e-1	3.8e-1	2.1e-1	8.1e-2
$k=2$	9.1e-1	8.4e-1	7.1e-1	5.4e-1	3.6e-1	2.8e-1
$k=3$	9.2e-1	8.7e-1	7.8e-1	6.4e-1	5.2e-1	4.7e-1
$k=4$	9.3e-1	9.0e-1	8.4e-1	7.4e-1	6.5e-1	6.0e-1
$k=5$	9.4e-1	9.1e-1	8.7e-1	8.0e-1	7.2e-1	6.9e-1
	$\alpha_b = 1, \beta_b = 1, \alpha_w = 1, \beta_w = 1$					
$k=1$	9.1e-1	8.3e-1	7.1e-1	5.0e-1	3.1e-1	2.3e-1
$k=2$	9.2e-1	8.7e-1	7.9e-1	6.3e-1	5.0e-1	4.3e-1
$k=3$	9.3e-1	9.0e-1	8.4e-1	7.4e-1	6.3e-1	5.8e-1
$k=4$	9.4e-1	9.1e-1	8.7e-1	8.0e-1	7.1e-1	6.7e-1
$k=5$	9.4e-1	9.2e-1	8.8e-1	8.2e-1	7.5e-1	7.2e-1
	$\alpha_b = 10, \beta_b = 0.1, \alpha_w = 1, \beta_w = 1$					
$k=1$	9.0e-1	8.4e-1	7.0e-1	4.9e-1	3.3e-1	2.8e-1
$k=2$	9.2e-1	8.9e-1	7.9e-1	6.4e-1	5.2e-1	4.7e-1
$k=3$	9.4e-1	9.1e-1	8.4e-1	7.4e-1	6.4e-1	6.0e-1
$k=4$	9.4e-1	9.1e-1	8.6e-1	8.0e-1	7.3e-1	6.8e-1
$k=5$	9.4e-1	9.2e-1	8.9e-1	8.2e-1	7.6e-1	7.4e-1
	$\alpha_b = 100, \beta_b = 0.01, \alpha_w = 1, \beta_w = 1$					
$k=1$	9.1e-1	8.4e-1	7.1e-1	5.1e-1	3.3e-1	2.9e-1
$k=2$	9.3e-1	8.9e-1	7.9e-1	6.5e-1	5.2e-1	4.8e-1
$k=3$	9.3e-1	9.1e-1	8.5e-1	7.4e-1	6.4e-1	6.0e-1
$k=4$	9.4e-1	9.2e-1	8.8e-1	8.0e-1	7.1e-1	6.9e-1
$k=5$	9.4e-1	9.3e-1	9.0e-1	8.4e-1	7.7e-1	7.5e-1

4.2 Nonconvex Domain

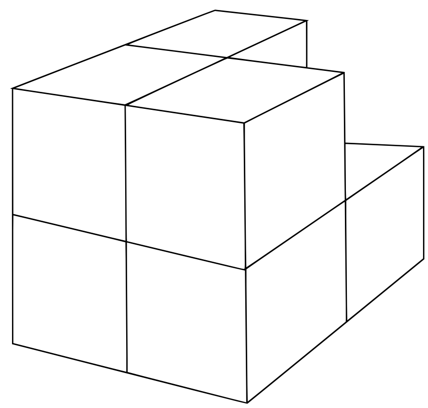

Fig. 2: Nonconvex domain

In the second numerical experiment we report the results for our model problem (1) on the nonconvex domain $\Omega = (0,1)^3 \setminus ([1/2,1]^3)$. We use the constant coefficients $\alpha = 1$ and $\beta = 1$ and other general settings are quite similar to those of Section 4.1. The results are presented in Table 2. It is observed that the method provides a uniform convergence of the V cycle multigrid. However, the contraction numbers are generally larger that those of the convex domain.

Table 2: Contraction numbers of the V-cycle multigrid method for the non-convex domain as in Fig. 2 with $\alpha = 1, \beta = 1$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$k=1$	9.3e-1	9.0e-1	8.2e-1	6.9e-1	5.5e-1	4.6e-1
$k=2$	9.5e-1	9.2e-1	8.5e-1	7.7e-1	6.8e-1	6.3e-1
$k=3$	9.6e-1	9.2e-1	8.8e-1	8.2e-1	7.7e-1	7.3e-1
$k=4$	9.6e-1	9.3e-1	8.9e-1	8.5e-1	8.0e-1	7.8e-1
$k=5$	9.6e-1	9.3e-1	9.0e-1	8.7e-1	8.4e-1	8.2e-1

References

1. D. N. Arnold, R. S. Falk, and R. Winther. Preconditioning discrete approximations of the Reissner-Mindlin plate model. *RAIRO Modél. Math. Anal. Numér.*, 31(4):517–557, 1997.
2. D. N. Arnold, R. S. Falk, and R. Winther. Preconditioning in $H(\mathrm{div})$ and applications. *Math. Comp.*, 66(219):957–984, 1997.

3. D. N. Arnold, R. S. Falk, and R. Winther. Multigrid preconditioning in $H(\mathrm{div})$ on non-convex polygons. *Comput. Appl. Math.*, 17(3):303–315, 1998.
4. D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$. *Numer. Math.*, 85(2):197–217, 2000.
5. S. C. Brenner and D.-S. Oh. Multigrid methods for $H(\mathrm{div})$ in three dimensions with nonoverlapping domain decomposition smoothers. submitted.
6. Z. Cai, R. D. Lazarov, T. A. Manteuffel, and S. F. McCormick. First-order system least squares for second-order partial differential equations: Part I. *SIAM J. Numer. Anal.*, 31(6):1785–1799, 1994.
7. V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.
8. R. Hiptmair. Multigrid method for $H(\mathrm{div})$ in three dimensions. *Electron. Trans. Numer. Anal.*, 6(Dec.):133–152, 1997. Special issue on multilevel methods (Copper Mountain, CO, 1997).
9. R. Hiptmair and A. Toselli. Overlapping and multilevel Schwarz methods for vector valued elliptic problems in three dimensions. In *Parallel solution of partial differential equations (Minneapolis, MN, 1997)*, volume 120 of *IMA Vol. Math. Appl.*, pages 181–208. Springer, New York, 2000.
10. T. V. Kolev and P. S. Vassilevski. Parallel auxiliary space AMG solver for $H(\mathrm{div})$ problems. *SIAM J. Sci. Comput.*, 34(6):A3079–A3098, 2012.
11. K.-A. Mardal and R. Winther. Preconditioning discretizations of systems of partial differential equations. *Numer. Linear Algebra Appl.*, 18(1):1–40, 2011.
12. D.-S. Oh. An overlapping Schwarz algorithm for Raviart-Thomas vector fields with discontinuous coefficients. *SIAM J. Numer. Anal.*, 51(1):297–321, 2013.
13. D.-S. Oh, O. B. Widlund, S. Zampini, and C. R. Dohrmann. BDDC algorithms with deluxe scaling and adaptive selection of primal constraints for Raviart–Thomas vector fields. *Math. Comp.* published online, available at https://doi.org/10.1090/mcom/3254.
14. P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.
15. P. S. Vassilevski and U. Villa. A block-diagonal algebraic multigrid preconditioner for the Brinkman problem. *SIAM J. Sci. Comput.*, 35(5):S3–S17, 2013.
16. B. I. Wohlmuth, A. Toselli, and O. B. Widlund. An iterative substructuring method for Raviart-Thomas vector fields in three dimensions. *SIAM J. Numer. Anal.*, 37(5):1657–1676 (electronic), 2000.